THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
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THE PSYCHOLOGY OF MATHEMATICAL ABILITY.*

A REVIEW OF THE PRESENT POSITION.

BY A. G. HUGHES, M.Ed., B.Sc.

BOTH terms in the title of this paper are probably suspect. The mathematician, versed in a subject with hundreds of years of noble tradition, tends to be suspicious of a paper on psychology; the teacher of mathematics, immersed daily in psychological relationships with ordinary children, tends to be suspicious of a paper on mathematical ability. The discussion promises to be unsatisfactory, to the one because psychology lacks mathematical precision, to the other because mathematical ability seems to present no urgent psychological problems. The teacher-mathematician may then be pardoned if he reacts sceptically to our title with the petition, "Psychology, if there be a science of psychology, help me to cope with mathematical disability." But these suspicions are not well founded. Not only is psychology, as we shall see, becoming increasingly mathematical, but also, if we interpret our terms as mathematicians should, disability is seen to be merely the negative aspect of ability. As such, it will not be excluded from our discussion.

The first problem is whether there is such a thing as special mathematical ability, positive or negative. That is to say, is the ability to do mathematics something different from the ability which enables a pupil to do science or to do composition or to "do" his teacher? Is it possible for a boy of good general intelligence to be deficient in mathematical ability, or for a brilliant mathematical student to be lacking in all-round mental efficiency? There are two ways of getting an answer to such questions. The first is by considering the opinions of teachers, the sum of which we may call collective common sense; the second is by considering the findings of science which with Huxley we may call organised common sense.

Opinion is not unanimous. On the one hand, there is a tendency, especially perhaps among mathematicians, to assume that ability to do mathematics is in itself an indication of a mind of high quality; and in school, at least, mathematics is looked upon as a reliable test of intelligence. But, on the other hand, intelligent pupils do fail in matriculation mathematics; there are honours graduates who were forced to take specialist courses to avoid the mathematics of a general course; there have been great men like Kant, and

* A paper read before the Mathematical Association (London Branch) on February 25, 1928. It has been partly rewritten to meet some of the points raised in the subsequent discussion, and I have also had the benefit of some very helpful suggestions and criticisms from Professor Cyril Burt. In acknowledging my indebtedness to Professor Burt I ought to state that he does not necessarily subscribe to all the speculations I have advanced in this paper

Goethe, and Emerson, who experienced much difficulty in learning mathematics.

We turn then to science, and we find that psychologists have already attacked the problem by the method of measurement. It may be objected that to attempt to measure before we know the exact nature of what we are measuring is absurd. But it is a method commonly used by scientists with success; attempts to measure heat, electricity and intelligence preceded and facilitated any clear understanding of their nature. J. B. S. Haldane has pointed out how the clue to an understanding of thunderstorms and fevers came from measuring lengths of mercury columns in glass tubes; then came a stage of innumerable graphs and tables of figures; "and out of this intellectual mess there suddenly crystallises a new and easily grasped idea." * So we start with rough ideas that mathematical ability is connected with success in school mathematics and with inborn general intelligence, and we proceed to measure the degree of this success and of this intelligence. For both purposes we substitute the new methods of the psychologist for the traditional methods of the teacher. In brief, the new examination contains many little questions requiring many short answers, whereas the old examination contained a few comprehensive questions requiring a few long answers. It is claimed that the new method is more scientific than the old. There being only one possible right answer to each question, anyone provided with the answers can mark the scripts, and the marks scored by any one script will always be the same no matter who does the marking. By requiring short answers only, it is possible to test one ability more or less pure; the results of a test in arithmetical problem-solving, for example, are not complicated by differences in ability to write quickly and legibly. †

The types of tests used to measure intelligence are now well known; ‡ a review of some typical tests used to measure mathematical ability may however be useful here. No single kind of test has been found sufficient to reveal such ability, but according to one investigator, § the one single test which has most prognostic value is one requiring the testee to interpolate missing numbers in a series, *e.g.*

$$2,\ 5,\ 10,\ 17,\ 26,\ 37,\ \ldots,\ \ldots,$$
$$1,\ 3,\ \ 4,\ 12,\ 13,\ 39,\ \ldots,\ \ldots.$$

To test arithmetic ability a battery of tests is compiled, including tests in straightforward adding, subtracting, multiplying, and dividing, and tests in problem-solving. ‖ In algebra, tests involving the manipulation of mechanical processes are given, and tests involving the use of general symbols. For example, 14 questions of the following type have been set to be done in five minutes:

50 articles cost 4/-. How many shillings do n articles cost?
A is 3 times as old as B. If B's age is now x years, what will A's age be n years hence?

Ballard's test in algebra ¶ contains 100 questions to be done in two hours. Another type of algebra test is one requiring inference with symbols, the testee being required to fill in the gaps, *e.g.*

$$a < b = c;\quad \therefore\ a \quad c.$$
$$a = b \not< c = d;\quad \therefore\ a \quad d.$$

* J. B. S. Haldane, *Possible Worlds*, p. 139.

† For a fuller defence of the "atomised" test, see P. B. Ballard, *The New Examiner*, Chapter VI.

‡ See, for example: P. B. Ballard, *Group Tests of Intelligence*; Cyril Burt, *Northumberland Standardized Tests, General Intelligence.*

§ A. L. Rogers, *Experimental Tests of Mathematical Ability and their Prognostic Value* (1923).

‖ See, for example, Cyril Burt, *Northumberland Standardized Tests, Arithmetic.*

¶ P. B. Ballard, *The New Examiner*, pp. 200-202.

Many ingenious tests of ability in geometry have been devised of which the following are examples:

1. Ten shapes are given and the testee is required to show by drawing construction lines how they could be constructed by correctly arranging 3 out of 4 given shapes.

2. Drawings of a hand are given in 49 different positions and the testee is required to say in each case whether the hand is left or right.

3. 11 rectilinear figures of different shapes are given and lettered *A*, *B*, *C*, etc. 19 unlettered shapes are also given, and the testee is required to write in the appropriate letter, if any of the lettered shapes would fit them without being turned over.

4. The pupil is required to fill in the omitted steps in reasoning in geometrical exercises. The omissions are indicated by dotted lines.

5. A cube of three-inch edge and painted red all over its surface is shown. The pupils are asked to imagine the cube cut into smaller cubes each of one inch edge. They then write down the number of small cubes that would have no sides, 1 side, 2 sides, 3 sides painted red after the three-inch cube has been cut as stated.

Having administered and marked the tests, the next important step is the interpretation of the results. Here the modern psychologist calls in the aid of the mathematician, whose argument, reduced to its lowest terms, is somewhat as follows. If the attainments in all the mathematical tests were due to a special factor, mathematical ability, and to no other factor, the position obtained by any one child for one test would resemble closely the position obtained by the same child for any other mathematical test. For example, if a boy were well above the average in the cube test, he would, within the limits of experimental errors, be equally well above the average in the algebra tests, in the arithmetic tests, and in all the others. Of course this is a purely hypothetical case, for though two tests may involve a common ability, it never happens that this single ability is the only one.

The enquiry then resolves itself into one of measuring the degree of similarity between the results of the various tests. Theoretically, this similarity can be complete as in the imaginary case just cited; or there can be complete dissimilarity, in which case the boy who was top for, say, arithmetic problems would be bottom for, say, the "hand" test, and so on, the one order-of-merit list being the exact reverse of the other. Or the similarity might be of any degree between these two extremes. The exact degree can be measured by the statistical figure known as the coefficient of correlation. Its value may vary from $+1$ for complete similarity to -1 for complete dissimilarity.

We shall only have to deal with coefficients of values between 0 and $+1$, for it is found that between the results for any two tests whatever involving mental work, there is nearly always some positive resemblance. Now this inevitable resemblance between the results of all mental tests introduces a complication into our problem of measuring mathematical ability; for, when we find the correlation which exists between the results of our mathematical tests, we shall not know how much of the resemblance is due to the factor or factors common to all mental tests, and how much, if any, is due to a factor peculiar to tests of mathematics, that is, to mathematical ability as such. Here again we need the aid of the mathematician.* He notices that the intercorrelations between mental tests can be arranged in a hierarchical system, and he points out that this result can be most simply explained on the hypothesis that the various intercorrelations are mainly brought about by the original correlation of each test with a single factor common to all the tests. An example will make this clear. Let us assume the existence of such a common

* In the explanation of the Two Factor Theory which follows, I am very deeply indebted to the expositions of Professor C. Spearman in his book, *The Abilities of Man*, and of Professor Cyril Burt in his pamphlet, *The Measurement of Mental Capacities*.

factor, G,* and let us suppose that four tests, A, B, C, D, are correlated with G to the extent of 0·9, 0·8, 0·7, 0·6 respectively. It can be proved that if G is the only factor operating to produce these correlations, then the correlation of any one test, say A, with any other test, say B, will be the product of their respective correlations with G, *i.e.* $rA \,.\, B = (0{\cdot}9 \times 0{\cdot}8) = 0{\cdot}72$. We thus obtain the following hierarchical arrangement of the intercorrelations between the four tests:

Test				A	B	C	D
Correlation with G				0·9	0·8	0·7	0·6
A	0·9			—	0·72	0·63	0·54
B	0·8			0·72	—	0·56	0·48
C	0·7			0·63	0·56	—	0·42
D	0·6			0·54	0·48	0·42	—

It will be noticed that if we consider any four coefficients forming a rectangle, the difference between the products of diagonally opposite pairs must be equal to zero, *e.g.* in the enclosed rectangle, $(0{\cdot}63 \times 0{\cdot}48) - (0{\cdot}56 \times 0{\cdot}54) = 0$. Such a difference is called a tetrad difference. If therefore the tetrad differences are, within the limits of their probable errors, equal to zero, the intercorrelations are most probably due to the operation of a single general factor.† The abilities required for tests A, B, C, and D are composed of two factors, one common to all, the other specific to each. Diagrammatically, they may be represented thus:

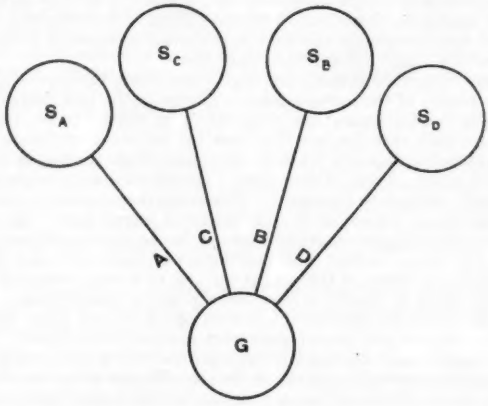

FIG. 1.

Now, if on trying out the various tests, two of them, say A and C, correlate more highly than the operation of one common factor would explain, say 0·90 instead of 0·63, then the tetrad difference would be:

$$(0{\cdot}90 \times 0{\cdot}48) - (0{\cdot}56 \times 0{\cdot}54) = 0{\cdot}13.$$

This would indicate that the abilities required for tests A and C comprise not

* It may be roughly identified with what is popularly called "general intelligence."

† This criterion is due to Professor C. Spearman. He gives a mathematical proof of it in his recent book, *The Abilities of Man*, Appendix.

only the general common factor but also a special common factor.* That is to say, the specific factors involved in tests A and C overlap thus:

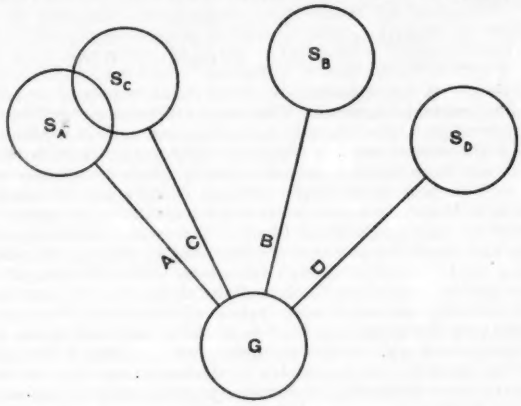

FIG. 2.

This is exactly the type of result obtained by Collar;† tests A and C were arithmetic tests, one involving computation, the other problem-solving; tests B and D were non-mathematical tests of intelligence. This result suggests that there is a special ability, over and above that due to general intelligence, by virtue of which a pupil can do both computative and problem work in arithmetic.‡ Similar results have been published by other investigators using tests involving algebra and arithmetic.§

The next question investigated by Collar was this: Does arithmetical ability fall into two main divisions, ability in computation and a higher ability involved in problem-solving? The intercorrelations between the arithmetic tests seemed to indicate that this was so, but when the correlation due to G was eliminated by the method of partial correlation, the tetrad differences became insignificant. Collar concluded therefore that computation and problem-solving involve in addition to G one common overlapping factor; this factor we may call special arithmetic ability. It is the only special factor entering into both computation and problem-solving; we may conclude therefore that ability to solve arithmetic problems differs from ability to calculate, only in so far as it involves more G. Thus a person who can compute but who cannot learn higher algebra, is probably deficient in G; there is no special "higher mathematics" ability which he may lack. But if a person can do higher algebra but cannot compute, we conclude, not that he is deficient in G or in a special computative capacity, but that he has probably lost interest in the base art of computation.

* The existence of special common factors can also be shown by the method of partial correlation. As Spearman points out, the tetrad difference criterion can be easily derived from Yule's formula for partial correlation. See Spearman, *op. cit.* pp. ii and iii, Appendix.

† D. Collar, "A Statistical Survey of Arithmetical Ability," *Brit. Jnl. Psych.* xi. 135-158 (1920).

‡ Brown and Burt had previously called attention to special arithmetical ability. See W. Brown, "An Objective Study of Mathematical Intelligence," *Biometrika*, vii. (1910); "Some Experimental Results in the Correlation of Mental Abilities," *Brit. Jnl. Psych.* iii. (1910); C. Burt, *The Distribution and Relations of Educational Abilities*, p. 58 (1917).

§ A. L. Rogers, *op. cit.*; W. S. Flack, "Investigation of Mathematical Ability in the Classroom," *Forum of Education*, iv. 44-56 (1926); L. Fouracre, "Psychological Tests of Mathematical Ability," *Forum of Education*, iv. 201-205 (1926). See also C. Spearman, *op. cit.* pp. 230-232.

The next question is whether the special ability so far discovered is also operative in geometric work. The following is a tetrad obtained from the work of Rogers: *

	Arith.	G_a
Geometry - - -	0·249	0·367
G_b - - - -	0·424	0·561

Geometric ability was measured by three tests: superposition, symmetry, and matching solids and surfaces. The tetrad difference is −0·015 (P.E. 0·035). Such a result suggests that the only factor common both to arithmetic and to geometry is the general one—intelligence. Spearman concludes that "there appears no real basis for the common opinion which would take arithmetic and geometry to furnish one single ability. Their union as 'mathematics' seems rather to be merely one of practical convenience." In contradistinction to this view we may quote Branford: † "The science (of mathematics) is a true unity and should be so developed from the beginning, and Geometry is the uniting fluid that runs through the whole." But for the psychologist, arithmetic and geometry are fundamentally different; the one deals with numerical relations, the other with spatial relations. As Whitehead ‡ says, "The 'spaciness' of space and the 'numerosity' of number are essentially different things, and must be directly apprehended. None of the applications of algebra to geometry or of geometry to algebra go any step on the road to obliterate this vital distinction." With very young children, geometry begins with sheer delight in shapes as shapes; arithmetic and geometry are quite distinct. Though later, geometry makes use of arithmetic, the two subjects do not form a "true unity" any more than physics and arithmetic do, although some problems of physics are most conveniently expressed in arithmetical terms. The word "geometry" in its etymological sense of earth-measurement is inadequate; it suggests merely the one aspect of the "practical" stage of the subject which is most prominent in the early history of geometry as we know it. It contains no suggestion of the practical use to which shapes can be put in designing, nor any suggestion of the "romance" and "generalisation" stages which respectively precede and follow the "practical" stage. Geometry, like most other aspects of science, has much in common with arithmetic and algebra, but at no one of the three stages of their progress is either subject completely absorbed in the other. A clear realisation of the independent and continuous development of geometry as the science which deals with space relations would do much to improve current methods of geometry teaching. §

One further question has been investigated by statistical methods. Is there a special ability which enables us to do geometry? The evidence quoted by Spearman ‖ is mainly negative, but in a more recent research, ¶ Fouracre found "a pronounced correlation between the geometry tests" after the factor of general intelligence had been eliminated by partial correlation. He concluded therefore that there is a special geometric ability.

We see then that three ideas on mathematical ability have crystallised clearly out of the "intellectual mess" of correlation tables. (1) Special arithmetic ability does exist; (2) apart from general intelligence, it is the only common arithmetic ability, and it permeates all arithmetic and algebraic work from lowly computation to the higher regions of problem-solving; (3) it is not an ability which enters into work in geometry. A fourth idea has been

* Quoted from C. Spearman, *op. cit.* p. 232.

† B. Branford, *A Study of Mathematical Education*, p. 123.

‡ A. N. Whitehead, *An Introduction to Mathematics*, p. 240.

§ See T. P. Nunn, "An Elementary School Syllabus in Mathematics," *Forum of Education*, ii. (1924).

‖ *Op. cit.* p. 228.

¶ L. Fouracre, *op. cit.* p. 203.

suggested, but it needs confirmation; it is (4) that a special geometric ability exists, independent of general intelligence and of arithmetic ability.

These four ideas at once suggest others, all of which will doubtless in time be tested by scientific measurements mathematically interpreted. Special arithmetic ability, appearing as it does in computative work, seems to be connected with a "preference" for working with algebraic and number symbols as distinct from verbal symbols; it may be an ability to perceive and organise numerical relations, but it does not appear to be a special kind of mathematical intelligence, differing from or transcending general intelligence. There is no doubt that mathematical reasoning, dealing with concepts of a high degree of abstractness, with symbols which are removed not only one but often two steps from reality, demands a high degree of *G*, but there is no experimental evidence that it differs in kind from reasoning in general. Rignano comes to precisely the same conclusion by a very different route, namely, by a psychological analysis of mathematical reasoning. All reasoning, he concludes, entails the performance of mental experiments, and possesses the same fundamental characteristics.*

The small amount of experimental work on ability in geometry, and the limited nature of the tests so far used, do not warrant any similar conclusion being drawn in this branch of mathematics; though Rignano's analysis suggests that if the existence of special geometric ability be confirmed, it will be found to be a factor common to simple perception of space relations and to higher work involving geometric reasoning. That is to say, special geometic ability will appear to be connected with a "preference" for working with shapes and perhaps solids,† as distinct from working with verbal and literal symbols; it may be an ability to perceive and organise spatial relations, and as such be essentially dependent on vision and visualization. To solve these problems, it appears that future investigators in this subject ought to devise tests differing from tests of superposition and symmetry as algebraic problems differ from arithmetic computation.

We may next inquire into the origin of these special abilities or "preferences." Are they inherited or acquired? When an intelligent boy fails in mathematics, is it possible that he is suffering from a lack of inborn mathematical capacities, or has he merely failed to make a mathematical use of general capacity? The latter view has been boldly championed by Dr. V. Hazlitt;‡ she believes that we inherit general capacity in varying degrees, and that all special abilities, even musical and literary abilities, for example, are developed by special acquired interests which divert this general power into special channels. She admits that special temperamental and physical characteristics may also play their part. This view, she points out, agrees with that of Dr. Johnson when he said that "the true genius is the man of large general powers accidentally determined to some particular direction." But, to borrow an illustration from Burt, it is difficult to believe that in a different environment "Beethoven might have invented the phonograph and Edison have written the symphony in C minor."§ It may be that what we are inclined to call inborn special capacity is the result of a special combination of general characteristics of three kinds, namely, intellectual, temperamental, and physical. With some few persons, the equipoise may be so nice that the slightest appropriate stimulus starts a special interest which, rapidly gaining

* The psychology of mathematical reasoning is therefore not discussed in this paper; we are immediately concerned with the special factors which differentiate mathematical ability from other forms of ability. Readers who wish to follow up the role of general intelligence in mathematical work will find a good introduction to a fairly extensive literature on the subject in *The Psychology of Reasoning*, chapters vii. viii. and ix. by E. Rignano.

† The "collective common sense" of teachers seems to suggest that ability to do solid geometry may be distinct from ability to do plane geometry.

‡ V. Hazlitt, *Ability* (1927).

§ C. Burt, *The Measurement of Mental Capacities*, p. 41.

force, breaks down all obstacles. With most persons, the equipoise is probably more stable, and any exceptional special ability is developed only by the operation of strong and steady environmental influences. But a child is not the mere sport of such influences; within the limits of his powers he can resist them or he can make the most of them. The degree of this opposition or co-operation will depend largely upon the nature of the child's emotional attitude towards the subject, and in the formation of such attitudes, experiences in early childhood are of prime importance. Over and over again, investigators have found one of the chief causes of mathematical disability to be irrational fear or dislike of the subject. Some misunderstanding, some feeling of hopeless incapacity perhaps induced by premature cramming, and the child early develops an inferiority complex. In some cases the pupil carries over to the subject an antipathy he feels towards the teacher, but in either case the result is the same, namely, an inhibition of interest caused by an aversion from the subject. The problem is often still further complicated by the fact that pupils may be entirely unaware of the real nature of such aversions. Dr. Ernest Jones records that he has had patients who, after having had such inhibitions removed by psycho-analysis, have developed "an unsuspected capacity for the previously impossible subject of study." * As teachers of mathematics we should, particularly in the early stages, be very careful, by making haste slowly, by giving an abundance of practical work, and by maintaining an encouraging attitude, to do all in our power to prevent a child getting the idea that he cannot do mathematics. This is obviously of great importance when dealing with highly emotional children of good intelligence, for though intellectually they are potential mathematicians, emotionally they are very prone to prejudice.

On the more purely intellectual side, the early environment can assist in laying the foundation for mathematical ability by giving the child an abundance of experience with the materials of mathematics, numbers and shapes. In arithmetic we must beware of introducing the child to the number symbols before he has had the requisite experience with the number realities; in algebra the problem is similar, and though the novelty of playing with literal symbols is generally sufficient to carry most pupils successfully through the first few chapters of symbol-juggling, many of them soon lose interest in doing what is apparently meaningless, and so they join the ranks of the mathematically disabled. As Rignano † has explained, if the fundamental nature of reasoning is the making of a mental experiment, it is necessary for the reasoner to consider symbols as symbols, that is, he must be confident that they represent always some physical reality.‡ The resistance offered, even by eminent mathematicians, to the so-called imaginary numbers so long as no one had succeeded in giving them a real significance, is a striking illustration of the type of resistance offered by school children to meaningless symbol-juggling; it is a natural protest against bad teaching though it is generally dubbed "stupidity." Turning to geometry, we must beware lest our legitimate anxiety to see that junior-school children master the essentials of arithmetic leads us to neglect giving them the necessary geometric experiences. A total neglect of geometry in junior schools is not uncommon, and herein doubtless lies one reason why many pupils, who at age eleven show promise in arithmetic, fail when they come to study the broader subject of mathematics in secondary schools. Intimate familiarity with numbers and with shapes both plane and solid, acquired in early years by pleasant playway methods, is

* E. Jones, *Psycho-Analysis*, p. 648.

† E. Rignano, *op. cit.* p. 159.

‡ This is perhaps not so true of the advanced pure mathematician, for, as Santayana says, "It is a pleasant surprise to him and an added problem if he finds that the arts can use his calculations, or that the senses can verify them, much as if a composer found that the sailors could heave better when singing his songs." G. Santayana, "Revolutions in Science," *The New Adelphi*, March 1928.

probably the surest guarantee that general capacity will later be diverted into mathematical channels.

It is interesting to note that poverty or mean surroundings is no bar to the acquirement of familiarity with the materials of mathematics. In fact, poverty may induce a special interest in numbers, and a lack of books may drive a child to activities which lay a good foundation for both algebra and geometry. We should therefore expect intelligent scholarship children from poor uncultured homes to display mathematical ability more often than linguistic ability. The fact that calculating prodigies have generally been born in a miserable milieu seems to confirm this view.

It may be objected that the fact that mathematics seems to run in families * suggests that mathematical capacity as such is inherited to a greater degree than is suggested in this paper. But this is not necessarily so, for a child born of mathematical parents is more likely to have a very favourable environment for the development of mathematical interests than one born of, say, literary parents. The same is true of a boy who goes to a school with a strong mathematical tradition. In determining special interests, no influence is so inevitable or so relentless as tradition. It seems probable, for example, that much apparent incapacity for mathematics among women † is due to the fact that it has been traditional for girls to take little interest in the subject; this explanation is confirmed by recent developments in feminine interests and abilities along many different lines. Girls are not inferior to boys when judged by intelligence tests or by tests in computation, but they are not the equal of boys in mathematical problem-solving. This may be partly accounted for by the influence of a wrong tradition, but it may also be that the problems used in school appeal more strongly to boyish interests than to girlish interests. This is well exemplified in the very excellent book of problems devised by Professor T. P. Nunn in connection with his book on *The Teaching of Algebra*. Perhaps it is inevitable that mathematics should make a weaker appeal to most girls than to most boys; on the other hand, it may be that we are only waiting for an ingenious and inspiring woman teacher of mathematics to complete Professor Nunn's work.

On the whole, therefore, I incline to the view that special mathematical ability is more of the nature of an acquired preference than of an inherited capacity, though some intellectual, temperamental, and physical factors, or combination of them, may well predispose certain persons to acquire such preferences. Just as a person with deformed fingers is not likely to become an expert pianist, so a person suffering from uncorrected astigmatism is not likely to achieve distinction in geometry. A child of an introverted type of temperament is less likely to be keenly observant of the shapes and forms of the concrete things around him that an extrovert. If the introvert happens to have little power of manipulating his visual images, he is then very unlikely to develop a preference for geometry, though if he has the necessary intelligence he may become a keen algebraist and later may take a keen delight in analytic geometry. Speaking generally, an introverted type of temperament seems to be more favourable to the development of mathematical ability of either kind than an extroverted type. Even the geometer as a rule appears to be fundamentally an introvert. It is true he uses facts of the concrete world as a basis on which to build his imaginative structure, but as Aldous Huxley suggests, this may be a compromise by which the introvert adjusts his native tendencies to our present social tradition which is predominantly extroverted.‡ It is

* Fehr, by a questionnaire sent out to 100 eminent mathematicians, found that in two-thirds of his cases mathematical ability was in the family. See W. Brown, *Mind and Personality*, chapter ix. "Mind and Mathematical Ability."

† On the subject of sex-differences in mathematical ability, see C. Burt, *Mental and Scholastic Tests*, pp. 295-302 and 403-406; A. E. Cameron, "A Comparative Study of the Mathematical Ability of Boys and Girls in Secondary Schools," *Brit. Jnl. Psych.* xvi. 29-49 (1925).

‡ Aldous Huxley, *Proper Studies*, pp. 78 ff.

probably a difference in effectiveness of visual imagery rather than a difference in kind of temperament which distinguishes the geometer from the analytic type of mathematician.

Another inborn factor which may influence the development of mathematical ability is emotional instability. The unstable child is very often specifically backward in arithmetic. Such a child seems to need a concrete medium to help him to focus attention; he is not able to attend steadily enough to perceive and organise abstract relations.

But, though a lucky combination of inborn favourable factors may lead directly to the development of exceptional mathematical ability, and though exceptional mathematical disability may result from an unlucky combination of inborn unfavourable factors, I believe that for the very great majority general intelligence of all the inborn characteristics is the only important *sine qua non* of mathematical ability. Further, I suggest that an interest in mathematics in both of its main divisions is part of every child's natural interest in the objective world; it is common to both sexes, and generally makes its appearance as early as the second year of a child's life. The interest may possibly be stronger among boys than among girls. It certainly is so in regard to mechanical contrivances, though the differences between the play of little boys and little girls in this respect may be due in part to imitation; in the real world, men and not women have in the past tended machines. On the other hand, the early age at which such differences appear suggests an inborn sex difference. If it exists, it is probably due to temperamental causes. The non-aggressive emotions, tenderness, submissiveness, and fear, are more pronounced in girls; the aggressive emotions, domination, assertiveness, and anger, are more pronounced in boys. Whereas a girl will turn a cylinder into a doll which she can nurse, a boy will make it into a steam-roller which he can drive; while the girl is manipulating the clothes and talking to the dressed-up cylinder, the boy is getting experience with the naked cylinder and is relatively taciturn. We are tempted to suggest that here lies at least one key to an understanding, not only of boys' superiority in mechanical ability,* but also of their relative inferiority in linguistic ability and in manual ability necessitating fine finger work. But whatever be the cause, boys appear to be superior to girls in mechanical ability, and this superiority may not be without its influence on mathematical ability both in geometry and algebra.

The general conclusion to be drawn from our discussion is that if pupils are more backward in mathematics than is warranted by their general intelligence, we ought to suspect that something has gone wrong with the development of their fundamental interests in number and shape. It is not safe to assume complacently that such pupils are by nature lacking in mathematical capacity; nor should we, in the absence of a careful examination of their temperamental characteristics, assume that their lack of interest in mathematics is natural. The fault most probably lies in their education, using the term in its broadest sense. How far parents are to blame for what happened in pre-school days, how far the infant school is to blame, how far the primary school or the secondary school—these are some of the problems which still remain for teachers and psychologists to solve. Psychology has been merciful in showing us that we teachers are not responsible for the manufacture of general intelligence; that we must leave to nature and the eugenists. Not so, however, with the manufacture of mathematical ability; psychology suggests that this is a responsibility which in a very large measure we must shoulder.

A. G. HUGHES.

Prof. Lodge, Dr. Sheppard, Prof. Levy and Mr. Katz took part in the subsequent discussion on Mr. Hughes' paper.

* See M. McFarlane, "A Study of Practical Ability," *Brit. Jnl. Psych. Monograph Supplement*, viii. (1925); Cyril Burt, *The Measurement of Mental Capacities*, p. 41; C. Spearman, *op. cit.* p. 229.

Prof. Lodge stressed a distinction between the logical faculty involved in Geometrical Study and the Practical Geometry which appeared in Mr. Hughes' remarks as the material of the psychologist experiments, and Dr. Sheppard continued this line of thought.

Prof. Levy pointed out that Mr. Hughes' paper dealt with the measurement rather than the psychology of mathematical ability, which the psychologists were measuring without knowing what it was. It would be possible to define mathematical ability when a meaning could be given to the words "it is obvious."

Mr. Hughes accepted Prof. Levy's point and suggested that by his endeavours to measure mathematical ability the psychologist might be making progress to elucidating what it is.

Mr. Katz thought that the experiments of the psychologist atomised mathematical ability; problems that were a real test were more complex than those used by the psychologist.

Mr. Hughes replied that it was only by simplifying the investigation that quantitative results could be obtained.

GLEANINGS FAR AND NEAR.

585. A reviewer of Griffin's *Mensuration* in the *Athenæum*, 1852, p. 1269, observes that we owe this work "to the condescension of a Senior Wrangler."

586. Did you feel either of these earthquakes? They have made Whiston ten times madder than ever. He went to an ale-house at Mile End to see one who . . . had predicted the earthquakes. The man told him . . . that he had it from an Angel. Whiston rejected this as apocryphal. For he was well assured that if the . . . secret was to be communicated to anyone, it would be to himself.

587. Mrs. Buller's youngest sister was Mrs. Strachey. One of her husband's brothers had translated a curious old Hindoo treatise on Algebra which made his name familiar to me.—Carlyle, *Irving*, p. 23.

[Mrs. Buller was the mother of Charles Buller. Edward Strachey published *Bija Ganita: or the Algebra of the Hindoos.* 4to, 1813, a translation of Bhaskara's treatise.]

588. When a young scholar at Eton was recommended to Sir Henry Savile for a good wit, he replied: "Out upon him . . . give me the plodding student. If I would look for witts I would go to Newgate; there be the witts."

589. When Brinkley, the astronomer, was made an Irish bishop, Lord Norbury said that he might *thank his stars* for his elevation.

590. At Newark I observed the market place. In London we should call it a square, though its sides were neither straight nor parallel.—Dr. Johnson to Mrs. Thrale.

591. Dominie Sampson—"a profound scholar and mathematician."

592. The conjuror in *Guy Mannering* calculates the fatal day for Harry Bertram from a mere *view* of the heavenly bodies, *i.e.* had an eye that could measure angles in the heavens to 1″.

593. A man intent on squaring the circle will see squares in circles wherever he looks, and sometimes he will find them.—Sir Richard Burton, *Vikram and the Vampire.*

594. To Lamb from Paris. June 9, 1802.—. . . I have formed a *little* acquaintance with the *great* Carnot, whom I find very pleasant. He is one of the firstrate Mathematicians in France, *i.e.* in the world.—*Letters of Thomas Manning to Charles Lamb*, p. 77, ed. G. A. Anderson, 1925. [*per* Mr. F. Puryer White].

OPERATIONAL METHODS IN MATHEMATICAL PHYSICS.

By Prof. H. S. Carslaw, Sc.D.

§ 1. This essay-review of Jeffreys' very welcome and valuable Tract with the above title * has been written at the editor's request. Many readers of the *Gazette* must have heard of Heaviside's operational method of solving the equations of dynamics and mathematical physics. If they have tried to learn about them from Heaviside's own works, they have attempted a difficult task. Nothing more obscure than his mathematical writings is known to me. A Cambridge Tract is now at their disposal. From it much may be learned; but the air of mystery still—at least in part—remains.

In every course on differential equations the student learns to solve by a symbolical method the linear differential equation with constant coefficients. He writes the equation:

$$a_0\frac{d^nx}{dt^n}+a_1\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_nx=T,$$

as $f(D)x=T$, and he finds a particular integral with the help of the inverse operator.

Part of Jeffreys' Tract deals with an extension of this method. The solution of the linear differential equation with constant coefficients (or equations, if we are concerned with simultaneous equations) is found, satisfying the given initial conditions, and this without the trouble of finding the arbitrary constants of the complementary function and the relation between them. In this section there is no serious difficulty. But I wonder if the time saved in the solution of the equations is worth the labour involved in learning (or teaching) the new method.

The other, and more important, part of the Tract deals with something much more obscure, Heaviside's operational method. What this is will appear from an example taken from his writings,† the simple heat conduction problem, where a solid bounded by the plane $x=0$, and extending to infinity in the direction of the positive axis of x, has its plane surface kept at the constant temperature v_0, the initial temperature through the solid being zero.

In this problem we have to find v to satisfy:

$$\frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2}, \quad \text{when } x>0,\ t>0; \qquad \ldots\ldots\ldots 1.1$$

$$v=v_0, \quad \text{when } x=0,\ t>0, \qquad \ldots\ldots\ldots 1.2$$

$$v=0, \quad \text{when } x>0,\ t=0. \qquad \ldots\ldots\ldots 1.3$$

Putting $\dfrac{\partial}{\partial t}=p=kq^2$, we have

$$\frac{\partial^2 v}{\partial x^2}=q^2v;$$

and a suitable formal solution is

$$v=e^{-qx}v_0$$

$$=\left(1-qx+q^2\frac{x^2}{2!}-\ldots\right)v_0. \qquad \ldots\ldots\ldots 1.4$$

Now p stand for $\dfrac{\partial}{\partial t}$, and Heaviside found that if we interpret $p^{\frac{1}{2}}$, operating on 1, as $(\pi t)^{-\frac{1}{2}}$, and obtain $p^{\frac{3}{2}}$, $p^{\frac{5}{2}}$, ... from $p^{\frac{1}{2}}$ by differentiation, the symbolical

* *Cambridge Tracts in Mathematics and Mathematical Physics.* No. 23, by Harold Jeffreys, (Camb. Univ. Press), 1927. Price 6s. 6d. net.

† *Electromagnetic Theory*, by Oliver Heaviside, vol. 2, p. 13, 1899.

solution given in 1. 4, and similar solutions of other problems of the same kind, do really satisfy the equations from which we start.

In this case,* beginning with $p^{\frac{1}{2}} . 1 = \frac{1}{\sqrt{(\pi t)}}$, we have

$$p^{\frac{3}{2}} . 1 = p . p^{\frac{1}{2}} . 1 = \frac{\partial}{\partial t} \frac{1}{\sqrt{(\pi t)}} = -\frac{1}{2\sqrt{\pi}} t^{-\frac{3}{2}}$$

and $$p^{\frac{5}{2}} . 1 = p . p^{\frac{3}{2}} . 1 = \frac{1 . 3}{2 . 2} \frac{1}{\sqrt{\pi}} t^{-\frac{5}{2}}, \text{ etc.}$$

And p, p^2, etc., operating on 1, give zero.

Interpreting the symbols in 1. 4 in this way, we have

$$v = \left\{ 1 - \frac{2}{\sqrt{\pi}} \left(\frac{x}{2\sqrt{(kt)}} \right) - \frac{1}{3} \left(\frac{x}{2\sqrt{(kt)}} \right)^3 + \frac{1}{2!5} \left(\frac{x}{2\sqrt{(kt)}} \right)^5 - \ldots \right) \right\} v_0$$

$$= v_0 \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{(kt)}}} e^{-u^2} du \right\}$$

$$= v_0 \left(1 - \text{Erf} \frac{x}{2\sqrt{(kt)}} \right),$$

with the usual notation for the "error function,"

$$\text{Erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

§ 2. Heaviside himself hardly claimed that he had "proved" his operational method of solving these partial differential equations to be valid. With him † mathematics was of two kinds: *Rigorous* and *Physical*. The former was *Narrow*: the latter *Bold* and *Broad*. And the thing that mattered was that the Bold and Broad Mathematics got the results. "To have to stop to formulate rigorous demonstrations would put a stop to most physico-mathematical enquiries." Only the purist had to be sure of the validity of the processes employed.

Jeffreys (p. 47) agrees that the arguments upon which Heaviside relies are "in many cases suggestive rather than demonstrative." And he seems to think his Tract places the operational method on another plane. But, if I may say so, there is too much of the Bold and Broad School in the work. It leaves me still doubtful if it is wise to make this method one of the tools of the mathematical physicist. There is no room for mystery in mathematics. If we can be clear, let us be so. And for my part I consider the best way of attacking many of these questions is to use contour integrals. It is only in England and America that the mathematical physicist is afraid of the elementary theory of the functions of a complex variable required in this method. And surely he need not indulge this fear in Cambridge. To adopt the words of Heaviside, which Jeffreys takes for the motto of his Tract,—

Even Cambridge mathematicians deserve justice.

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* If p stands for $\frac{\partial}{\partial t}$, one would expect p^{-1} to denote integration. Then, for positive integral values of n, we would have

$$p^{-n} . 1 = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}.$$

If this final formula is to hold for fractional values of n, we would have

$$p^{-\frac{1}{2}} . 1 = \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} = 2\sqrt{\frac{t}{\pi}}.$$

And $p^{\frac{1}{2}} . 1 = p . p^{-\frac{1}{2}} . 1 = \frac{1}{\sqrt{(\pi t)}}$.

† Cf. *loc. cit.*, p. 4.

§ 3. Turn now to Chapter I. of the Tract. This deals with the solution of the equations:

$$\left.\begin{aligned} &e_{11}y_1+e_{12}y_2+\ldots+e_{1n}y_n=S_1\\ &e_{21}y_1+e_{22}y_2+\ldots+e_{2n}y_n=S_2\\ &\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\\ &e_{n1}y_1+e_{n2}y_2+\ldots+e_{nn}y_n=S_n \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots 3.\ 1$$

where the y's are dependent variables, x the independent variable, e_{rs} denotes $a_{rs}\frac{d}{dx}+b_{rs}$, where a_{rs} and b_{rs} are constants, and the S's are known functions of x. It is not assumed that $a_{rs}=a_{sr}$, $b_{rs}=b_{sr}$, but it is assumed that the determinant formed by the a's is not zero, and that when $x=0$, we have $y=u_1$, and so on, the u's being known constants.

Now let Q stand for the operation of integrating with regard to x from 0 to x, so that $Qy=\int_0^x y\,dx$.

Perform the operation Q on both sides of each equation in 3. 1.

Then we have
$$\begin{aligned} Qe_{rs}y_s&=\int_0^x\left(a_{rs}\frac{dy_s}{dx}+b_{rs}y_s\right)dx\\ &=a_{rs}(y_s-u_s)+b_{rs}Qy_s\\ &=f_{rs}y_s-a_{rs}u_s, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.\ 2 \end{aligned}$$

where
$$f_{rs}=a_{rs}+b_{rs}Q. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.\ 3$$

Thus the equations 3. 1 and the initial conditions are together equivalent to the equations:

$$\left.\begin{aligned} &f_{11}y_1+f_{12}y_2+\ldots+f_{1n}y_n=v_1+QS_1\\ &\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\\ &f_{n1}y_1+f_{n2}y_2+\ldots+f_{nn}y_n=v_n+QS_n \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots 3.\ 4$$

where $v_r=a_{r1}u_1+a_{r2}u_2+\ldots+a_{rn}u_n$.

Let Δ stand for the operational determinant formed by the f's,

i.e.
$$\Delta=\begin{vmatrix} f_{11}, & f_{12}, & \ldots, & f_{1n}\\ f_{21}, & f_{22}, & \ldots, & f_{2n}\\ \ldots\ldots\ldots & \ldots\ldots\ldots & & \ldots\ldots\ldots\\ \ldots\ldots\ldots & \ldots\ldots\ldots & & \ldots\ldots\ldots\\ f_{n1}, & f_{n2}, & \ldots, & f_{nn} \end{vmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots 3.\ 5$$

If Δ is expanded by the ordinary rules of algebra, and equal powers of Q collected, we obtain a polynomial in Q. The term independent of Q is the determinant formed by the a's, and, by hypothesis, this does not vanish. Let F_{rs} be the co-factor of f_{rs} in Δ. Then F_{rs} is also a polynomial in Q.

Operate on the first equation of 3. 4 with F_{1s}, on the second with F_{2s}, etc., and add. The only term in the sum which does not vanish is

$$\sum_{r=1}^{n}(F_{rs}f_{rs})\,y_s \quad \text{or} \quad \Delta y_s.$$

Therefore
$$\Delta y_s=\sum_{r=1}^{n}F_{rs}(v_r+QS_r), \quad \ldots\ldots\ldots\ldots\ldots\ldots 3.\ 6$$

And
$$\begin{aligned} y_s&=\frac{1}{\Delta}\left\{\sum_1^n F_{rs}(v_r+QS_r)\right\}\\ &=\frac{\phi(Q)}{\Delta(Q)}+\sum_{r=1}^{n}\frac{\psi_r(Q)}{\Delta Q}QS_r, \quad \ldots\ldots\ldots\ldots\ldots\ldots 3.\ 7 \end{aligned}$$

where $\phi(Q)$ and $\psi(Q)$ are polynomials in Q, whose degree is ordinarily one less than that of $\Delta(Q)$.

The formal solution of 3. 7 can now be treated in the same way as the particular integral of $f(D)\,y=X$ is found in elementary work.

The operator on the right can be expanded in ascending powers of Q, and evaluated term by term; of course on the understanding that, if this gives rise to an infinite series, the S's are such that this series converges.

Alternatively the expressions $\dfrac{\phi(Q)}{\Delta(Q)}$, etc., can be broken up into partial fractions. Since the determinant formed by the a's is not zero, we may denote it by A, and $\Delta(Q)$ takes the form $A(1-a_1Q)\,(1-a_2Q)\,\ldots\,(1-a_nQ)$ where the a's are ordinarily different.

A fraction of the type $\dfrac{1}{1-aQ}$, operating upon a constant L, will give a term Le^{ax}; and, operating upon QS, it will give a term $e^{ax}\int_0^x Se^{-ax}dx$.

The case of repeated roots a is discussed in the text. And the argument is throughout quite similar to the usual treatment of the equation $f(D)\,y=X$.

§ 4. If now, with Heaviside, we write p^{-1} for Q, the equations 3. 4 become

$$\left.\begin{aligned}&(a_{11}+b_{11}p^{-1})y_1+(a_{12}+b_{12}p^{-1})y_2+\ldots(a_{1n}+b_{1n}p^{-1})y_n=v_1+p^{-1}S_1\\&(a_{21}+b_{21}p^{-1})y_1+(a_{22}+b_{22}p^{-1})y_2+\ldots(a_{2n}+b_{2n}p^{-1})y_n=v_2+p^{-1}S_2\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&(a_{n1}+b_{n1}p^{-1})y_1+(a_{n2}+b_{n2}p^{-1})y_2+\ldots(a_{nn}+b_{nn}p^{-1})y_n=v_n+p^{-1}S_n\end{aligned}\right\}. \quad 4.1$$

Multiply throughout by p, as if it were a mere number. Then we have

$$\left.\begin{aligned}&(a_{11}p+b_{11})y_1+(a_{12}p+b_{12})y_2+\ldots+(a_{1n}p+b_{1n})y_n=pv_1+S_1\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&(a_{n1}p+b_{n1})y_1+(a_{n2}p+b_{n2})y_2+\ldots+(a_{nn}p+b_{nn})y_n=pv_{12}+S_n\end{aligned}\right\}. \quad \ldots\ldots 4.2$$

Solve the n equations of 4. 2 by ordinary algebra, and we obtain a solution identical with that of 3. 7, except that p^{-1} will take the place of Q, and both numerator and denominator will be multiplied by the same power of p.

We write this solution as

$$y_s=\frac{1}{\Delta}\left\{\sum_{r=1}^{n}F_{rs}(pv_r+S_r)\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots 4.3$$

where F_{rs} and the determinant Δ are not to be confused with the same operational symbols in 3. 6.

The operators on the right of 4. 3 are of the form $f(p)/F(p)$, where $f(p)$ and $F(p)$ are polynomials in p, $F(p)$ being of degree n and $f(p)$ of the same or lower degree. Resolving $F(p)$ into its n factors, supposed all different, we have the algebraical identity,

$$\frac{f(p)}{pF(p)}=\frac{f(0)}{F(0)p}+\sum_a\frac{f(a)}{aF'(a)}\frac{1}{p-a},$$

whence

$$\frac{f(p)}{F(p)}=\frac{f(0)}{F(0)}+\sum_a\frac{f(a)}{aF'(a)}\frac{p}{p-a}.$$

If this operates on unity, the term $\dfrac{p}{p-a}$ gives rise to e^{ax}, and*

$$\frac{f(p)}{F(p)}\cdot 1=\frac{f(0)}{F(0)}+\sum_a\frac{f(a)}{aF'(a)}e^{ax}. \quad \ldots\ldots\ldots\ldots\ldots\ldots 4.4$$

If it operates on $e^{\mu x}$, we replace $e^{\mu x}$ by $\dfrac{p}{p-\mu}\cdot 1$.

* This is usually known as Heaviside's Expansion Theorem, or the Partial Fraction Rule. Cf. *loc. cit.*, p. 127.

Then $\dfrac{f(p)}{F(p)} \cdot e^{\mu x} = \dfrac{pf(p)}{(p-\mu)F(p)} \cdot 1$

$$= p\left\{\frac{f(\mu)}{F(\mu)}\frac{1}{p-\mu} + \sum_a \frac{f(a)}{(a-\mu)F'(a)}\frac{1}{p-a}\right\} \cdot 1$$

$$= \frac{f(\mu)}{F(\mu)} e^{\mu x} + \sum_a \frac{f(a)}{(a-\mu)F'(a)} e^{ax}.$$

If S is expressed as a linear combination of exponentials, we can apply this rule to each separately. Thus the solution applies to practically all functions known to physics.

It will be seen that in this section we replace d/dx in the original equations of 3. 1 by p, and to the right of each equation add the result of dropping the b's on the left and replacing the y's by their initial values. Then we proceed to solve these equations by ordinary algebra, and interpret our result by certain simple rules.

In this argument, in which I have followed the Tract pretty closely, there does not seem any lack of rigour, though there is just a little mystery left about p, for it is only p^{-1} which has been defined. It is obvious that in this discussion p is not just the operation of differentiation, for in that case pv_r, etc., would vanish.

If the reader wishes a proof on other lines, he has in Chapter II. an independent discussion based on Bromwich's interpretation of a function of p as a contour integral.* His proof for these dynamical equations is completely satisfactory.

§ 5. This work can be at once extended to equations of a higher order by breaking them up into equations of the first order.

For example, take the equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X, \quad \text{.....5. 1}$$

where a, b are constants, and X is a function of x. The solution is also to satisfy $y = y_0$ and $\dfrac{dy}{dx} = y_1$, when $x = 0$.

Introduce a new variable z given by $z = \dfrac{dy}{dx}$.

Then 5. 1 is replaced by the two equations:

$$\left.\begin{aligned} \frac{dz}{dx} + az + by &= X \\ z - \frac{dy}{dx} &= 0 \end{aligned}\right\}. \quad \text{.....5. 2}$$

Operating on these with Q, we have

$$\left.\begin{aligned} (z - y_1) + aQz + bQy &= QX \\ Qz - (y - y_0) &= 0 \end{aligned}\right\}. \quad \text{.....5. 3}$$

Thus

$$\left.\begin{aligned} (p+a)z + by &= py_1 + X \\ z - py &= -py_0 \end{aligned}\right\}. \quad \text{.....5. 4}$$

Solving 5. 4 by algebra, we have

$$(p^2 + ap + b)y = (p^2 + ap)y_0 + py_1 + X,$$

and

$$y = \frac{(p^2 + ap)y_0 + py_1 + X}{p^2 + ap + b}. \quad \text{.....5. 5}$$

Let the roots of $p^2 + ap + b = 0$ be α and β.

* The revived interest in Heaviside's operational method is due chiefly to a paper by Bromwich on "Normal Co-ordinates in Dynamical Systems," *Proc. London Math. Soc.* (2), vol. 15, 1916, and to other papers of his in which the method is freely used.

Then from 5. 5,

$$y=p\left\{\frac{(p+a)y_0+y_1}{(p-a)(p-\beta)}\right\}+\frac{1}{(a-\beta)}\left\{\frac{1}{p-a}-\frac{1}{p-\beta}\right\}X.$$

Therefore

$$y=\frac{1}{a-\beta}\left[\left\{\frac{p}{p-a}(y_1-\beta y_0)-\frac{p}{p-\beta}(y_1-ay_0)\right\}+\left\{\frac{1}{p-a}-\frac{1}{p-\beta}\right\}X\right]. \quad ..5.6$$

Now
$$\frac{p}{p-a}\,.\,1=\frac{1}{1-aQ}\,.\,1=e^{ax}.$$

And
$$\frac{1}{p-a}X=\frac{Q}{1-aQ}X=e^{ax}\int_0^x Xe^{-ax}dx.$$

Thus

$$y=\frac{1}{(a-\beta)}\left[(y_1-\beta y_0)e^{ax}-(y_1-ay_0)e^{\beta x}+e^{ax}\int_0^x Xe^{-ax}dx-e^{\beta x}\int_0^x Xe^{-\beta x}dx\right]. \quad 5.7$$

Worked-out examples of the solution of equations of this type are given on p. 14 of the Tract. It will be seen that to solve

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=X,$$

subject to the initial conditions, we write down what Jeffreys calls the *subsidiary* equation

$$(p^2+ap+b)y=(p^2+ap)y_0+py_1+X,$$

and proceed thereafter according to quite simple rules. The work is certainly much shorter than when the usual method is followed.

§ 6. If Heaviside's operational method were simply that described and proved in §§ 3-4, it would not have been the cause of so much debate. But we have pointed out in § 1 that it was used by him in the treatment of the partial differential equations of mathematical physics. Indeed most of his researches in electric waves are carried out with its aid.

Jeffreys still leaves much that is mysterious in this connection. He deals in Chapters IV. and V. with the equations $\frac{\partial^2y}{\partial t^2}=c^2\frac{\partial^2y}{\partial x^2}$ of wave motion and $\frac{\partial v}{\partial t}=k\frac{\partial^2v}{\partial x^2}$ of heat conduction. In the case of the former he says (p. 41), "we are led by our previous rules to consider the subsidiary equation"

$$(\sigma^2-c^2p^2)y=\sigma^2f(x)+\sigma F(x),$$

where σ stands for $\frac{\partial}{\partial t}$, p for $\frac{\partial}{\partial x}$, and the initial values of y and $\frac{\partial y}{\partial t}$ are $f(x)$ and $F(x)$. And p. 53, which he devotes to the "proof," does not help us much.

Again in the case of the heat equation (p. 55), he writes down the subsidiary equation

$$\sigma v-k\frac{\partial^2v}{\partial x^2}=\sigma v_0,$$

where $\sigma=\frac{\partial}{\partial t}$, and $v=v_0$, when $t=0$. His justification of the operational method is eleven lines on p. 66.

Bromwich, on whose work he relies, admits that he has not given a complete proof of the validity of his solutions in the case of continuous systems. In his own words * all that he has done is to "establish an analogy" between the formulae he uses for the operational method in the solution of the partial differential equations and that which he has proved for the dynamical equations and discrete systems.

* Cf. *loc. cit.*, *Proc. London Math. Soc.* (2), vol. 15, 1916, p. 421. But see also pp. 438 *et seq.*

In this matter I think Jeffreys' Tract is subject to criticism. Heaviside was quite open about it. He belonged to the Bold and Broad School. Jeffreys, we take it, to be of the Cambridge School, in which Mathematics is Mathematics. It is true that he obtains correct solutions of the problems he discusses. But it seems to me that too little is said in justification of the method employed.

§ 7. I take as an example his solution of the problem of pp. 59 and 60 by the operational method, and in the next section I give the solution by the method of contour integrals and the standard path adopted by myself. His problem requires the solution of the following:

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2}, \text{ when } 0<x<l, \text{ and } t>0 \quad \text{.....7. 1}\\ \frac{\partial v}{\partial x} - hv &= 0, \text{ when } x=0, \text{ and } t>0 \quad \text{.....7. 2}\\ v &= v_0, \text{ when } x=l, \text{ and } t>0 \quad \text{.....7. 3}\\ v &= v_0, \text{ when } 0<x<l, \text{ and } t=0 \quad \text{.....7. 4} \end{aligned}\right\}$$

The equation 7. 1 is replaced by the subsidiary equation

$$\frac{\partial^2 v}{\partial x^2} - q^2 v = -q^2 v_0, \quad \text{.....7. 5}$$

where $\frac{\partial}{\partial t} = \sigma = kq^2$.

Then 7. 3 and 7. 5 suggest

$$v = (1 - A\sinh q(l-x))v_0,$$

and 7. 4 shows that A should satisfy

$$qA\cosh ql - hv_0(1 - A\sinh ql) = 0.$$

Thus we have the operational solution

$$v = \left(1 - \frac{h\sinh q(l-x)}{q\cosh ql + h\sinh ql}\right)v_0. \quad \text{.....7. 6}$$

The values of σ which satisfy $q\cosh ql + h\sinh ql = 0$ are real and negative, and one form of the solution is given by the Partial Fraction Rule of § 4, except that in this case the denominator has an infinite number of zeros.

Again, we may write 7. 6 as

$$v = \left[1 - \frac{he^{-qx}}{q+h}\left(1 - e^{-2q(l-x)}\right)\left(1 - \frac{q-h}{q+h}e^{-2ql} + \dots\right)\right]v_0. \quad \text{.....7. 7}$$

If the length l is great enough to make the terms in e^{-2ql}, etc., negligible, this reduces to the first two terms,

$$v = \left[1 - \frac{h}{q+h}e^{-qx}\right]v_0. \quad \text{.....7. 8}$$

Now $q = \sqrt{\frac{\sigma}{k}}$ and Bromwich's Rule (Ch. II. § 2. 1), with an obvious change in the notation, gives

$$\frac{e^{-qx}}{q+h} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\sqrt{k}}{\sqrt{\mu}+h\sqrt{k}}\exp\left(\mu t - x\sqrt{\frac{\mu}{k}}\right)\frac{d\mu}{\mu}.$$

Put $\lambda^2 = \mu$ and take the corresponding path in the λ-plane.

The solution is then found to be

$$v = v_0\left\{\operatorname{Erf}\frac{x}{2\sqrt{(kt)}} + \exp(h^2kt + hx)\left(1 - \operatorname{Erf}\frac{x+2hkt}{2\sqrt{(kt)}}\right)\right\}. \quad \text{.....7. 9}$$

§ 8. Now dropping all talk about operational methods and returning to ordinary mathematics and contour integrals, let us find the solution of the problem of § 7.

It is convenient in all these heat problems to have the initial temperature throughout the solid zero, so we put $v = v_0 + u$ in the equations 7. 1 to 7. 4.

Thus we have

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \text{ when } 0<x<l, \text{ and } t>0 \\ \frac{\partial u}{\partial x} - hu &= hv_0, \quad \text{when } x=0, \text{ and } t>0 \\ u &= 0, \quad \text{when } x=l, \text{ and } t>0 \\ u &= 0, \quad \text{when } 0<x<l, \text{ and } t=0 \end{aligned}\right\} \quad \begin{aligned} &\ldots\ldots\ldots\ldots 8.1 \\ &\ldots\ldots\ldots\ldots 8.2 \\ &\ldots\ldots\ldots\ldots 8.3 \\ &\ldots\ldots\ldots\ldots 8.4 \end{aligned}$$

Equations 8. 1 and 8. 3 are satisfied by

$$Ae^{-ka^2t}\sin a(l-x).$$

From 8. 2 we see that A should satisfy

$$A(a\cos al + h\sin al) = -hv_0. \quad \ldots\ldots\ldots\ldots 8.5$$

Then we take the standard path (P) * of Fig. 1 in the α-plane chosen so that at infinity on the right the argument of α lies between 0 and $\frac{1}{4}\pi$, and on the left between $\frac{3}{4}\pi$ and π.

The path (P) in the α-plane

FIG. 1.

It will be seen that, when we take the integral over this path (P),

$$u = -\frac{hv_0}{\iota\pi}\int e^{-ka^2t}\frac{\sin a(l-x)}{a\cos al + h\sin al}\frac{da}{a}, \quad \ldots\ldots\ldots\ldots 8.6$$

satisfies 8. 1, 8. 2, and 8. 3.

We have still to prove that this value of u vanishes when $t=0$; that is, we have to show that

$$\int_P \frac{\sin a(l-x)}{a\cos al + h\sin al}\frac{da}{a} \quad \ldots\ldots\ldots\ldots 8.7$$

is zero.

Take the closed circuit of Fig. 2, consisting of the path (P), and the arc of a circle, centre at the origin, lying above this path.

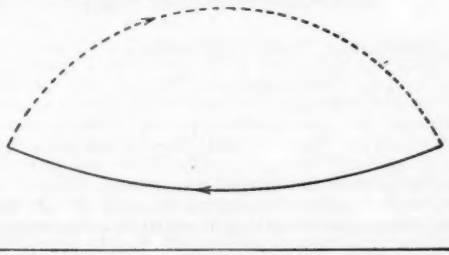

FIG. 2.

* In this section I follow the method and use the diagrams of Chapter XI. of my book on *Conduction of Heat* (Ed. 1921). See also Chapter X. §§ 80-90.

The roots of $a\cos al+h\sin al=0$ are infinite in number, and all lie on the real axis in the a-plane, to each positive root a_r corresponding a negative root $-a_r$.

There are thus no poles of the integrand of 8. 7 in the closed contour of Fig. 2, and the integral round it vanishes. But, when the radius of the circle tends to infinity, the integral over the circular arc vanishes.

It follows that

$$\int_P \frac{a\sin a(l-x)}{a\cos al+h\sin al}\frac{da}{a}$$

is zero.

Thus we have established that the function given by 8. 6 satisfies all the conditions of our problem.

We have now only to express this solution in real terms.

(i) As the integrand of 8. 6 is an odd function of a, on dividing by 2 we can replace the path (P) of Fig. 1 by the path (Q) of Fig. 3, since the integral over the dotted arcs vanishes at infinity.

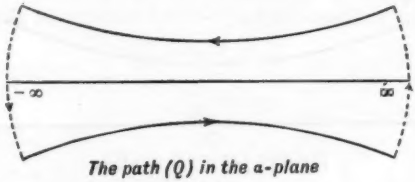

FIG. 3.

In this way we have, from 8. 6,

$$u=-\frac{hv_0}{2\iota\pi}\int_Q e^{-ka^2t}\frac{\sin a(l-x)}{a\cos al+h\sin al}\frac{da}{a}$$

$$=-hv_0\left[\frac{l-x}{1+hl}+2\sum_{r=1}^{\infty}\frac{\sin a_r(l-x)}{(1+hl)\cos a_rl-a_rl\sin a_rl}\frac{e^{-ka_r^2t}}{a_r}\right], \quad \text{.....8. 8}$$

by the Theory of Residues.

This result corresponds to that given by Heaviside's Partial Fraction Rule.

(ii) Again, we can write 8. 6 in the form

$$u=-\frac{hv_0}{\iota\pi}\int_P e^{-ka^2t}\frac{e^{\iota ax}}{h-\iota a}\frac{1-e^{2\iota a(l-x)}}{1+\frac{h+\iota a}{h-\iota a}e^{2\iota al}}\frac{da}{a}. \quad \text{.....8. 9}$$

For a first approximation, l being large, we have

$$u=-\frac{hv_0}{\iota\pi}\int_P e^{-ka^2t+\iota ax}\frac{da}{a(h-\iota a)}$$

$$=-\frac{v_0}{\iota\pi}\int_P e^{-ka^2t+\iota ax}\left(\frac{1}{a}-\frac{1}{a+\iota h}\right)da. \quad \text{.....8. 10}$$

We take the two parts of this integral separately.

For the first, as in Fig. 4, we can deform the path (P) into the real axis in the a-plane and a semicircle (vanishing in the limit) at the origin.

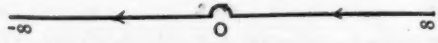

FIG. 4.

It follows that

$$\frac{1}{\iota\pi}\int_P e^{-ka^2t+\iota ax}\frac{da}{a}=1-\frac{2}{\pi}\int_0^\infty e^{-ka^2t}\frac{\sin ax}{a}\,da$$

$$=1-\operatorname{Erf}\frac{x}{2\sqrt{(kt)}}.* \quad \ldots\ldots\ldots\ldots\ldots\ldots 8.\,11$$

For the second, a similar deformation of the path (P) into the line $a+\iota h=0$ and a small semicircle at $(0,\ -\iota h)$, shows that

$$\frac{1}{\iota\pi}\int e^{-ka^2t+\iota ax}\frac{da}{a+\iota h}=e^{h^2kt+hx}\left(1-\frac{2}{\pi}\int_0^\infty e^{-k\beta^2t}\frac{\sin\beta(x+2hkt)}{\beta}\,d\beta\right)$$

$$=e^{h^2kt+hx}\left(1-\operatorname{Erf}\frac{x+2hkt}{2\sqrt{(kt)}}\right). \quad \ldots\ldots\ldots\ldots 8.\,12$$

Thus from 8. 10, 8. 11, 8. 12 :

$$v=u+v_0=v_0\left(\operatorname{Erf}\frac{x}{2\sqrt{(kt)}}+\exp(h^2kt+hx)\left(1-\operatorname{Erf}\frac{x+2hkt}{2\sqrt{(kt)}}\right)\right), \ \ldots. 8.\,13$$

as in 7. 9.

In Chapter XI. of my book on *Conduction of Heat* a number of examples will be found worked out in this way. The application to the general case follows when the Green's Function, expressed by similar contour integrals,† is used.

Sydney, Australia. H. S. CARSLAW.

The following notes from Dr. Jeffreys and Dr. Bromwich explain themselves :—

I think Prof. Carslaw has missed a point in his third paragraph, where he describes part of the tract as dealing with an extension of the usual method of finding a particular integral by means of the expansion of the inverse operator $1/f(D)$. The method I give is not an extension of this older method, but a substitute for it. Whereas in the older method the fundamental operator is $D\ (=d/dx)$, in mine it is Q or p^{-1}, which by definition means definite integration. p as such is not defined, because it never occurs in actual solutions. The principal advantage is the much greater generality of the method. Whereas it is the normal occurrence for an expansion of an operator in powers of Q to give an intelligible and unique answer, this happens with a series in powers of D only in freak cases where the series happens to terminate, or where the function operated on is a polynomial or an exponential, and special treatment is necessary in every type of application to show that the result actually satisfies the differential equation. In fact I think that no possible difficulty in dealing with the new method would excuse persisting in the obscurity of the old one, quite apart from the fact that it avoids the troublesome solution of the terminal conditions to determine the so-called arbitrary constants.

With regard to the comparative desirability of direct operational methods and complex integrals, I think they both have their spheres of application. It must be noticed that the operational method, when applied to sets of ordinary differential equations, is a special case of the method developed by Caqué, Fuchs, and Baker for linear equations even with variable coefficients, which is a practical method of the highest importance on its own account. It should be better known, partly because it gives a direct proof that solutions can actually be found to satisfy given terminal conditions, and partly because it tends to correct the idea of the average student that the solution of a differential equation consists necessarily and entirely in getting a formal answer in finite terms. I have the highest respect for complex integrals in their place, but their application to differential equations with variable

* Cf. my book on *Fourier's Series and Integrals* (Ed. 1921) Ex. 13 p. 195.

† Cf. *loc. cit.*, Chapter X

coefficients is very limited; and in such a problem as the motion of a stable dynamical system under given disturbing forces, when the forces are not analytic functions of the time, evaluation by direct integration is usually much easier than by means of complex integrals, because these require that the disturbance should be expressed as a Fourier integral first. On the other hand, complex integrals often form the most convenient way of evaluating an operational solution found by Heaviside's methods; whether it is better to introduce them for this purpose when the operational solution has been found, as Bromwich and I would do, or to bring them in at the very beginning, as Prof. Carslaw does, is a matter of taste and printing expenses. No logical question is involved. Prof. Carslaw's integral in his § 8, for instance, is obtained from that given by me by a simple change of variable; the justification of both is the same. I am sorry I took such a short space to show that the solutions I gave actually satisfied the differential equations of wave motion and heat conduction, but the proofs seem to be adequate even though short.

With regard to Heaviside's attitude to rigour, a point not usually mentioned is that when he used a new method of unknown validity he always checked it either by applying the method in other cases where the correct answer was known or by actual numerical computation. His methods may appear slipshod, but his precautions were such that they practically always gave the right answer. At any rate I have never heard of a wrong answer to a concrete problem in any of his works. He left mathematicians with the problem of explaining why the answers were right, which was duly solved, in time, by Bromwich; and it was then found that many of Heaviside's solutions could be obtained easily by workers without his amazing skill in manipulation, but using the theory of the complex variable.

I should not myself claim to be a member of either the Bold and Broad school or the Rigorous one. I am a theoretical physicist doing my best for the advancement of certain branches of physics. But I cannot see Prof. Carslaw's objection. I find it difficult to believe that he wants every stage of every proof developed to the last ϵ when any reader with the usual technique could do it quite well for himself, or that he really dislikes fundamentally the practice of getting the answer first and verifying it afterwards.

But the application of the methods to continuous systems involves another question quite independent of the mathematical one. The actual systems we are concerned with are not continuous, but have a finite number of degrees of freedom comparable with the number of atoms present. For a system of finite freedom the justification of the method is complete. The discrete system is the reality, the continuous one the approximation, and the question at issue is how closely the solution for it approximates to that appropriate to the actual system. Whether this solution actually satisfies exactly the approximate partial differential equation and terminal conditions that we impose on the continuous system is not part of the real question. The answer is not easy; I have tried to give one in relation to waves in strings,* but it is only one instance of the general question.

HAROLD JEFFREYS.

NOTE ON PROF. CARSLAW'S PAPER.

THE Editor of the *Mathematical Gazette* has kindly allowed me to see an advance proof of Prof. Carslaw's paper, and I take the opportunity of making some comments both on that paper and on Dr. Jeffreys' *Tract*.

In the first place, I am glad to see that Prof. Carslaw has come a little nearer to my own views as to the value of Heaviside's methods; we have had many discussions (both verbal and by letter) on the subject; and, in particular, I strongly urged a greater use of Heaviside's methods in the second edition of Prof. Carslaw's book.

* *Proc. Camb. Phil. Soc.* 23, 1927, 768-778.

Secondly, in the interests of historical accuracy I should like to put in a claim to have been the first worker to use the method mentioned in § 4 of the above paper (first stated in my own 1916 paper quoted there). Heaviside had used similar methods frequently in solving special problems, of the type most common in examples of a natural kind: but he never seems to have taken the trouble to formulate a general process for *arbitrary initial conditions*.

To illustrate the difference we may take the easily visualised illustration of a vibrating string: one of the most interesting examples is that of *the plucked string*, where a point P of the string is held aside and then released, as soon as the string is at rest. Heaviside would solve this by supposing a force F applied (at $t=0$) to the undisturbed string at the point P, where F is chosen so that it will just hold the string in equilibrium in the given initial state.

All that my method claims to do (beyond Heaviside's) is that it gives a simple rule for calculating the necessary forces (such as F) in rather more complicated problems. The method is very simple, when the result is known, but I spent some eight years thinking over the problem (from time to time) before obtaining the final version. In practice I have rarely found examples to which Heaviside's rule cannot be applied as readily as my own method.

Thirdly, I think that in § 6 Prof. Carslaw hardly does justice to Heaviside's methods: it is true that in 1914 (when the work was more or less completed for my paper published in 1916) I felt that I had only "*established an analogy.*" But my experiences during the subsequent fourteen years would justify me in stating that Heaviside's method has never led me astray—except in that kind of mistake which is due to human fallibility.* What further justification Prof. Carslaw asks for, I hardly understand.

Finally, with reference to § 8 of Prof. Carslaw's paper, I pointed out † in 1920 that Prof. Carslaw's standard paths follow at once from the complex integrals used in my 1916 paper (when modified to suit the problems of Conduction of Heat). Naturally Prof. Carslaw is at full liberty to regard his own paths as more fundamental because they were used by him in Heat problems before 1916; but it is much harder to adapt his paths to general dynamical problems, whereas (as I have shewn) they are themselves deducible from the paths used in my original paper.

To conclude, I may claim to have studied Heaviside's writings as closely as any reader, probably more closely than most. In speaking informally (November 1927) to a meeting of the Trinity Mathematical Society (Cambridge), I expressed the opinion that Heaviside must be ranked with the greatest constructive mathematicians of the nineteenth century; in many ways his work resembled that of Stokes, whose contributious to Pure Mathematics will, I hope, some day be valued more highly than they are by present-day writers of textbooks on Modern Analysis. Had Heaviside been able to make full use of Cauchy's methods of complex integration, then (to quote a well-known saying) "*we should have learned something*"; what is really marvellous is that a self-taught mathematician should have outstripped so far the finest Senior Wranglers of the Mathematical Tripos in what Dr. Jeffreys (see above) has rightly termed "*his amazing skill in manipulation*". An illustration of this skill may be found in Heaviside's solution of a modification of Lord Kelvin's problem on the cooling of the Earth: this extension was apparently solved by Heaviside in 1894, but his own actual solution has not been published so far as I know.‡

As to the value of Heaviside's methods, it seems not out of place to refer briefly to a letter from the late Lord Rayleigh. The latter had published (in

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* Compare the remarks in Dr. Jeffreys' preface to his *Tract*.

† *Proc. Camb. Phil. Soc.* vol. 20, 1921, p. 411. In particular this paper gives a justification for Heaviside's work quoted in § 1 of Prof. Carslaw's paper.

‡ The original statement was in the *Electrician* (14th Decr., 1894), and it is reprinted in the *Electromagnetic Theory*, vol. 2 (1899) § 230, with the remark that the solution "would frighten

the *Philosophical Magazine*) some interesting formulae dealing with the problem of "*Random Flights*"; and I noticed that the integrals used there could be readily turned into operational forms, and that certain formulae could be easily established in this way. Lord Rayleigh expressed considerable interest in this application of Heaviside's methods; and he took the opportunity to state that (in his opinion) these methods had not received adequate recognition; he also regretted that he had not been able himself to do more to support Heaviside's claims, in view of the fact that the operational method had been severely criticised from the side of Pure Mathematics. Thus it may also be of interest to refer to a conversation with Prof. G. H. Hardy, some time before 1914, in which he expressed the view that what Analysis then needed was a "*twentieth-century Euler*", capable of trying daring experiments with what one might call "*conjuring tricks in mathematics*"; the *E*-details of justification might then be filled in by other workers at their leisure. Probably Prof. Hardy regarded his later discovery of Ramanujan as having filled this *rôle* to some extent, at any rate in one region of analytical development; but I venture to suggest that (at least in his own domain) Heaviside was certainly a "*nineteenth-century Euler*".

As regards Heaviside's criticisms of Cambridge mathematicians (see, for instance, §§ 224, 237 of vol. 2 of his *Electromagnetic Theory*), I venture to suggest that they are of much the same character as Rudyard Kipling's remarks in "*Stellenbosh*":

> "We were sugared about by the old men,
> ('Eavy-sterned amateur old men)
> That 'amper and 'inder an' scold men
> For fear o' Stellenbosh!"

But in Heaviside's private letters to me, he often expressed regret at not having had the opportunity of studying at Cambridge in his younger days.

T. J. I'A. BROMWICH.

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595. Stray Thoughts of an Examiner.—Has the "aria" of a rectangle any relation to the music of the spheres?—W. F. S.

596. "The centre of gravity of three equal masses lies at their mid points. Their vertex lies on the medium through *A*." [*N.B.*—*A* was not otherwise defined.]

"A horse power is the work an imaginary horse can do in an hour."

"Mass is a big lump of stuff which has no shape about it." [Per Mr. R. Stoneley.]

597. Dec. 16, 1664.—Being Friday night, a blasing star was seene by some of University College. . . . It had a tayle (to their seeing) as long as 6 or 8 yards. . . . By another hand from New College . . . to be as big as a large hand with a tayle of 2 or three yards long.

May 16, 1668.—Between 9 and 10 of the clock at night, being an hour or two after supper there was seen by M. H. and A. W., and those of the family of Borstall, a *Draco Volans* fall from the sky. It made the place so light for a time, that a man might see to read. It seemed to A. W. to be as long as Allsaints steeple in Oxon. being long and narrow: and when it came to the lower region, it vanished into sparkles, and, as some say, gave a report.—*Athenae Oxonienses*, vol. i. Life of Anthony Ward, 1848 (ed. P. Bliss).

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timid readers, and perhaps some Cambridge mathematicians as well." The sequel is rather amusing; some ten years later this challenge was taken up by the examiners for Part II of the Mathematical Tripos, who printed a question, purporting to solve Heaviside's extension. But unluckily the answer given was wrong; as far as I know the correct solution was found first by me (using operational methods) and afterwards confirmed by Prof. Carslaw (using his own methods). These solutions are both given in the *Proc. Camb. Phil. Soc.* vol. 20, 1921.

EASY MATHEMATICS AND LAWN TENNIS.

By T. J. I'A. Bromwich, F.R.S.

In view of the popular appeal of Lawn Tennis, and of the fact that a large number of professional mathematicians depend upon this game for much of their recreation, it seems strange that no one has pointed out that a good deal may be learned (as to suitable tactics) by applying a few quite simple principles of dynamics and geometry to the game.

The following remarks refer to the game as played in a men's double: * after the age of thirty-five the players usually cease to take very great interest in a strenuous men's single, and (at about the same age) less anxiety to play in mixed doubles will be found. And once the ideas have been suggested which govern these tactics in a double, any one really interested will have little difficulty in working out similar rules for singles, or for mixed doubles, where it must be borne in mind that few ladies will volley with any real confidence.

In the first place, any student of geometry will have no difficulty in seeing that for a pair to cover the court (with as little trouble to themselves as possible) *they should adopt a formation in which the line joining the players is parallel to the net* (or nearly so); and again it is easy to see that *the most really vulnerable area is* BETWEEN *the two players, rather than the areas which are left outside the players* (that is, between each player and the nearer side-line).

These are, of course, general principles; *they cannot be of universal application*, and they must be modified in the light of the known methods of play and the special personality of individual opponents. For this reason closeness to the net has the further advantage (apart from the mathematical reasons given below) that the probable play of the attackers can be studied more easily, and their strokes (to some extent) anticipated in consequence of the knowledge so gained.

The court is taken as a rectangle 80 † feet by 32 feet, and the net as 3 feet high.

§ 1. Elementary Dynamics of the Lawn Tennis Court.

Although the actual flight of the ball must be greatly affected by air-resistance, yet we can get an idea of the general character of the problem by using the familiar parabolic trajectory.‡ Take then the height of the net as h, and suppose that a ball is to be hit at a height h_0 so as to skim the net and just fall into the court; we take the distances from the net to be x, l respectively.

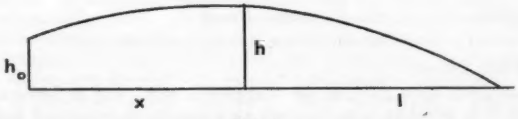

FIG. 1.

Then if u is the horizontal component of velocity we easily find that

$$\frac{h-h_0}{x}+\frac{h}{l}=\frac{g}{2u^2}(x+l)=\frac{\frac{1}{2}gt^2}{x+l}$$

where t is the whole time of flight.

* The players are supposed also to be *right-handed*: the necessary modifications due to one or two left-handed players can be easily introduced, but no new ideas are involved.

† Accurately 78 feet; but exact figures are not very valuable here.

‡ In actual play, the flight is modified a good deal by "cut" and "top-spin." The former is a very important feature in Court-Tennis, and consequently figured largely in the early days of lawn-tennis—in modern good fast play, "cut" is almost non-existent. "Top-spin" on the other hand may be seen often, but its effect is too complicated to be considered here.

From the point of view of the driver, it is of very great importance to make t *small* (so as to give his opponent little time to anticipate the stroke); and accordingly, h_0 being partly at his disposal, we have the FIRST WORKING RULE: *Take the ball as high above the ground as possible.*

To consider the effect of position, we may write the formula as

$$\tfrac{1}{2}gt^2=(h-h_0)\left(1+\frac{l}{x}\right)+h\left(1+\frac{x}{l}\right),$$

and in the subsequent discussion there is one fundamental difference, according as h_0 is *greater* or *less* than h: for definiteness we may distinguish these alternatives as the *smash* and the *drive*. The latter may be considered first.

The Drive.

Here $h_0 < h$; and thus $(h-h_0)/h$ is *positive*, but it will be supposed *fairly small*, say about $\frac{1}{4}$; that is, *the height h_0 will be $\frac{3}{4}h$* ($=2$ ft. 3 in.). Thus:

$$\frac{\tfrac{1}{2}gt^2}{h}=\tfrac{1}{4}\left(1+\frac{l}{x}\right)+\left(1+\frac{x}{l}\right),$$

and so the value of t is *least* for $x=\frac{1}{2}l$, and then is t_0 where

$$t_0^2=\tfrac{3}{16}\left(\tfrac{9}{4}\right)=\tfrac{27}{64}, \text{ or } t_0=\tfrac{1}{8}(5{\cdot}2)={\cdot}65 \text{ sec.}$$

This corresponds to a distance of some 20 feet from the net, if the driver aims at the back-line; but in practice it is usual to come up to, say, 10 or 12 feet, and to aim at a point perhaps 30 feet beyond the net. The value is then

$$t_1^2=\tfrac{3}{16}\left(\tfrac{7}{3}\right), \text{ or } t_1={\cdot}66 \text{ sec.},$$

but the slight increase in time for *this* stroke is more than compensated by the gain to be found in smashes. The speed u is about 60 ft./sec. in the second position and nearly 90 ft./sec. in the former.

*The Smash.**

Here $h_0 > h$; to have a clear idea of the magnitudes involved, we shall suppose h_0 to be from 4 to 5 feet; and so an average value for $(h-h_0)/h$ would be $-\frac{1}{4}$. It is now evident that x *must be as small as possible* (to reduce t as far as may be). But as x is decreased, it is evident that the negative term (the first) in the formula will overpower the positive term: this simply indicates the fact (familiar in all *net-play*) that *the net is no longer any real obstacle.* Then the racket usually directs the ball slightly *downwards*: but to make an easier calculation, let us suppose that the stroke is *horizontal*, and then we have the two simple formulae:

$$\tfrac{1}{2}gt'^2=h_0, \quad ut'=x+l.$$

These give $t'={\cdot}5$ (for $h_0=4$ feet)

and $u=80$ ft./sec. (taking $x+l=40$ feet).

The Base-line Drive.

Here we shall have $x=l$, at least approxinately, and so $\frac{1}{2}gt^2=2(2h-h_0)$; nowadays it is usual to take the ball at a height of perhaps 2 feet, so that again t is about ·7, but *the necessary speed is considerably greater*, and would seem to be at least 100 ft./sec. Since the energy varies as the square of the initial speed, *the amount of energy used in a base-line drive is about three times as much as from a point* 10 *feet from the net.*

§ 2. GEOMETRY APPLIED TO THE QUESTION OF POSITION IN THE COURT.

If the opponent is driving from the base-line (unless he possesses exceptional skill) he is almost forced to aim at the base-line (of course a *lob* can be dropped

* In what is usually called "*smashing*" the ball will be hit at a height of 6 or 7 feet: here we have used the phrase *smash* as a conveniently short phrase to distinguish the two types of stroke.

on the side-lines, but lobs must be treated on other principles). Thus, at a distance of 10 feet from the net, a length of $(\frac{40+10}{80})\ 32=20$ feet is exposed to fire: thus each player is responsible for 10 feet to be covered. It is a matter of common experience that few players can aim quite accurately at the corners of the court, so that probably not more than 9 feet needs to be covered by each player, but there is more need to cover the inside than the

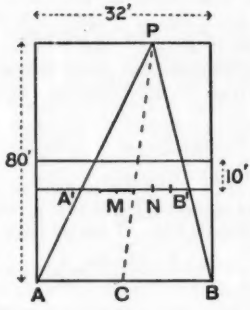

FIG. 2.

outside. The length to be covered by the *left-hand* player has been thickened in the figure. It is to be carefully noticed that, *in order to provide the best cover, the players (represented by M, N) should move to their* RIGHT, *following the position of the driver at P.*

The players should, therefore, try to form mental pictures of the area into which driving is possible, so as to place themselves to the best possible advantage along $A'B'$. Once this idea is grasped, the mind quickly indicates suitable points for defence; of course, in trying to defend against a smash, the problem is far more complicated, since *then* cross-shots are comparatively easy. Still the same principles can be used as a guide.

It is not difficult to apply similar considerations to *the position of the server's partner*: he must first decide, after observing the pitch of the service, whether there is any real danger of being passed at the side-line. When the pitch is near the centre-line, a passing shot is so difficult that this risk is almost negligible; then the player will be well-advised to move towards the centre-line of his court (so as to help the server to cover the danger zone). But if the pitch is near the side-line, a passing-shot is quite likely, and the player must remain as close to the side-line as is necessary to cover it; for the server will be fully engaged in trying to cover the rest of the court. As to the server himself, in either case his (further) side-line will be in little danger, and his best policy is to come up towards the centre-line, keeping a watch on his partner, so as to cover up any possible gap between them.

POSTSCRIPT.

ABOUT three weeks after putting down the above formulae I was asked by a lady player if there was an explanation (by mathematics) of a statement made by Señorita Alvarez to the effect that *if a ball is* ABOVE *the shoulder it may be hit at* ANY *speed, without being driven out of court, provided that the ball* GRAZES THE NET.

This statement can easily be justified by the same formulae; but it is a little easier to write down first some general formulae (from which our earlier formulae were found in the first place). The parabolic trajectory is used (as already explained); then suppose that u, v are the horizontal and vertical

components of the initial velocity, and that t_0 is the time to the net. Then we have

$$x = ut_0, \qquad x+a = ut,$$
$$h_0 - h = vt_0 + \tfrac{1}{2}gt_0^2, \qquad h_0 = vt + \tfrac{1}{2}gt^2.$$

Thus
$$\frac{h_0-h}{x} = \frac{v}{u} + \tfrac{1}{2}gt_0, \quad \frac{h}{a} = \frac{v}{u} + \tfrac{1}{2}g(t_0+t),$$

and so
$$\frac{h}{a} - \frac{h_0-h}{x} = \tfrac{1}{2}gt = \frac{g}{2u}(x+a).$$

From this formula those used previously will follow instantly.

For our immediate purpose we note that $(h_0-h)/x$ and also $\frac{1}{2}gt$ are both positive.

Thus:
$$\frac{h}{a} > \frac{h_0-h}{x} \quad \text{or} \quad \frac{x}{a} > \frac{h_0-h}{h}.$$

Accordingly, then, *if* $h_0 \geqq 2h$ *(i.e. 6 ft.), no ball can possibly be driven out.*

In the play of a person of moderate height it is perhaps reasonable to reckon "*shoulder-height*" at about 5 ft. 6 in. Then we have

$$\frac{h_0-h}{h} = \frac{66-36}{36} = \frac{5}{6},$$

and so $(a-x) < \frac{1}{6}a$.

This means that, *unless the driver is within 6 feet (say) of the base-line, a ball at the level of the head cannot be driven out, however hard the ball is hit (keeping it down to the net).*

T. J. I'A. BROMWICH.

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598. As to indexes, the art of making them seems to be lost: very few books, except those on law, have anything deserving the name. The reason is obvious enough: where proper information is not given, the table shows the spot where it ought to be, *vacant*; and a good index points out not only what *is* in the work, but also that which *is not*.—De Morgan, *Table*, P.C. xxiii. 496.

599. Will it be believed that there is at least one *Greek* word in the Latin versions (of Euclid) and in Simson, which is *not* in the original. That which we call an "axiom," which will make our readers give Euclid credit for the particular use of the word '*ἀξίωμα*' is by him called '*κοίνη ἔννοία*,' two common words implying "that which is in the understanding of every man."—De Morgan, *Q.J.E.* ii. p. 247.

600. Solid geometry is seldom or never taught before plane trigonometry, that is, a purely conventional arrangement has placed a very easy part of the subject after one of much greater difficulty; so that, in fact, access to the easier part is practically forbidden to all who do not first master the harder. . . . We should like to know to how many mathematical teachers *per cent.* it has occurred, instead of drawing one plane perpendicular to another on paper, to fold the paper itself, and place the two folds at the required angle.—De Morgan, *Q.J.E.* ii. p. 250.

601. Flogging works on a principle recommended by physicians, of curing disorder in a part which cannot be got at, by producing the same in another part which can.—De Morgan, *Q.J.E.* ii. p. 267.

602. Your lordship says that space is not extended. It will also follow that saying infinite space, or immensity, is an attribute of the Deity, is not the same with saying that he is infinitely extended, which would indeed be a most improper extension.—Richard Price to Lord Monboddo, Dec. 11, 1780.

MATHEMATICAL NOTES.

906. [D. 6. c. e.] *Euler Numbers.*

It must have struck everyone as strange that the Euler numbers $E_1, E_2, \ldots$ in the expansion

$$\sec x = 1 + E_1\frac{x^2}{2!} + E_2\frac{x^4}{4!} + E_3\frac{x^6}{6!} + \ldots,$$

viz. $E_1=1$, $E_2=5$, $E_3=61$, $E_4=1385$, ... end in 1 and 5 alternately. A complicated proof of this and related facts is given by Stern in *Crelle's Journal*, Bd. 79 (1875). He also refers to a paper by Sylvester (*Comptes Rendus*, T. 52, p. 212) which I have not been able to consult. The following simple and obvious proof has the advantage of entirely removing the mystery.

Let $\quad y=s=\sec x,\ t=\tan x, \quad y_1=dy/dx,\ y_2=dy_1/dx, \ldots,$

then $\quad y_1=st,\ y_2=s(1+2t^2), \quad y_3=s(5t+6t^3),$

$$y_4=s(5+28t^2+24t^4), \quad y_5=s(61t+180t^3+120t^5).$$

Also $E_n=$ value of y_{2n} when $x=0$.

But $$y_{2n+4}-y_{2n}=\frac{d^{2n-1}}{dx^{2n-1}}(y_5-y_1).$$

The coefficients of y_5-y_1 are all divisible by 60; the coefficients of y_n, when it is written in the above form are all integers, by their method of formation; hence the coefficients of $y_{2n+4}-y_{2n}$ are divisible by 60, since a constant factor can be removed to the outside of a differential operator. In particular $E_{n+2}-E_n\equiv 0 \pmod{60}$; this is Stern's result.

We can apply the same method to the coefficients of

$$y=\tan x = x + T_1\frac{x^3}{3!} + T_2\frac{x^5}{5!} + \ldots$$

and show that $T_{n+2}-T_n\equiv 0 \pmod{30}$. As a matter of fact, $T_{n+2}-T_n\equiv 0 \pmod{90}$, but this method does not seem capable of giving this result.

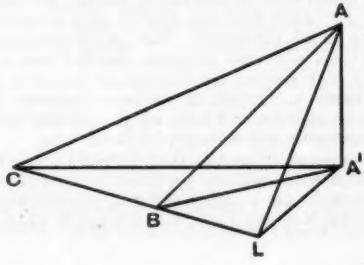
H. G. F.

907. [P. 1. f.] *Orthogonal Projection of a Plane Area.*

The elementary theorem that the orthogonal projection of a plane area is equal to the product of the area by the cosine of the angle between its plane and the plane of projection is one which, in my experience, tends to be "taken as read," or, at any rate, to receive somewhat scant attention in spite of its frequent and important applications in Applied Mathematics.

The proof, commonly given, based upon the method of cutting strips from the figure and its projection by planes perpendicular to the axis of projection, leaves much to be desired if the figure is rectilineal. The purpose of this note is to show that a quite accurate proof of the theorem for the case of any polygon, convex or re-entrant, can be given from first principles.

Case i. Consider first a triangle ABC projected on to a plane through a side BC. If $A'BC$ be the orthogonal projection and AL be drawn perpen-



dicular to BC, it follows that $A'L$ is also perpendicular to BC and ALA' is the angle between the planes $= \theta$ (say).

Hence area $A'BC$: area $ABC = A'L : AL = \cos\theta$.

Case ii. Next consider a plane of projection which cuts the sides BC, CA, AB at X, Y, Z respectively. We may regard the triangle ABC as

$$AYZ + BZX - CXY$$

and apply Case i. when the result follows at once.

Case iii. Any polygon, convex or re-entrant, can be considered as made up of triangles, so that the theorem follows for any polygon.

Case iv. We may extend to the case of a plane figure having a *convex* curvilinear boundary by considering the orthogonal projections of a closely inscribed polygon and the circumscribed polygon formed by tangents to the curved boundary at the vertices of the inscribed polygon. The area being always comprised between the areas of these two polygons, the theorem in this case follows on passing to the limit when the perimeter of either polygon tends to the curved boundary.

Case v. Finally, for any closed curvilinear boundary, we may divide the figure up by chords joining consecutive points of inflexion and apply Cases iv. and iii. as required.

E. H. SMART.

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603. Chief Baron Pollock (Senior Wrangler, 1806) to his son, Sir Frederick Pollock, when an undergraduate at Trinity: . . . I knew less of mathematics than you do when I went to college. Of algebra I knew literally nothing till about the June before I went to reside, but I steadily worked through all Bonnycastle. . . . I did not read the fifth book of Euclid till within a short time of taking my degree. Had it been thrust on me by compulsion when I began mathematics I never should have been senior wrangler. . . . I consider (I. 4) to be much more of a "pons asinorum" than the 5th. Indeed I regard it as the test of a geometric spirit. If it be fairly conquered, it proves the existence of that logical accuracy which is the soul of mathematics, and to elicit and cultivate which is the great benefit which such studies confer as a branch of education [1833]. I. pp. 57-59.

My last paper (on the Theory of Numbers), as I learn from a letter of yesterday, is to appear in the *Philosophical Transactions*. This is very gratifying. I wrote it since I was 84, and I doubt whether there be any production of so ancient a person in the whole series. . . . [1868]. II. p. 187—*Personal Remembrances of Sir Frederick Pollock*, 1887.

604. Vieta's *Munimen adversus Nova Cyclometrica* was a refutation of Joseph Scaliger's asserted quadrature of the circle, though the name of Scaliger is not mentioned in it. This eminent scholar was exceedingly angry, and attacked Vieta with much bitterness. But he afterwards, according to De Thou, changed his tone, admitted his error, and did justice to his opponent. Vieta himself had a high respect for Scaliger, as might be inferred from his suppression of the name. . . . In one of Casaubon's letters to De Thou . . . he says, that on one occasion he and a friend paid a visit to Vieta, and that, Scaliger's name coming up in conversation, Vieta said, "I have so great an admiration of that astounding genius, that I should think he alone perfectly understands all mathematical writers, particularly those of the Greeks." And he added that he thought more highly of Scaliger when wrong than of many others when right.—De Morgan, *Vieta*, P.C. 26, pp. 315-316.

REVIEWS.

Interpolation. By J. F. STEFFENSEN. Pp. vii + 248. Price 36s. 1927. (The Williams & Wilkins Company. Baltimore: Baillière, Tindall & Cox, 8 Henrietta Street, Covent Garden, W.C. 2.)

Three generations of Danish writers have devoted special attention to the subject of interpolation. Oppermann (1817-1883) had an unrivalled knowledge of the work done by Newton, by Newton's great predecessor Briggs, and by Newton's pupils and successors, and he did much to stimulate interest in the subject not only in Denmark but in England.

Thiele (1838-1910), who derived his inspiration from Oppermann, published the *Theory of Observations* in English in 1903, but his most important work is *Interpolations-Rechnung* (Leipzig, 1909), in which he advocated the fundamental importance of the theory of divided differences, and pointed out how by their use differential coefficients could be introduced into a scheme of differences and into formulas derived from such a scheme.

Dr. Johann F. Steffensen is Professor of Actuarial Science at the University of Copenhagen, the post having been created for him and being personal to himself, so that it lapses at his death. This book on interpolation, which will we think prove to be a landmark in the history of the subject, owes its existence to the lectures which the author has in recent years given to actuarial students, and is with a few additions and simplifications a translation of the Danish edition published in 1925. It is a great and comprehensive treatise, rigorous in its methods, building up the subject from the very foundation, and dealing with interpolation formulas, construction of tables, inverse interpolation, summation formulas, the calculus of symbols and interpolation with several variables. The style is severely condensed, and the concise Σ notation is used for the statement of series, which accordingly are in a form with which English students are not yet very familiar. In reading the book the student may be recommended for his own instruction to write out the first few terms of the functions in the usual English way, and he will find as he grows accustomed to the author's methods that every chapter opens up new ground and supplies stimulating and suggestive ideas.

At the very outset the author remarks that it is insufficient to derive formulas of interpolation on the assumption that the function with which we have to deal is a polynomial of a certain degree; and he defines the function $f(x)$ with which he deals as a real single valued function of x, continuous in the range under consideration and possessing in that range a continuous differential coefficient of the highest order of which use is made. He shows in the course of his work that any such function can be represented as the sum of a polynomial of degree n, and a remainder-term which is of such a simple nature that if two numbers are known between which the differential coefficient of order $n+1$ is situated we can find limits to the error committed by neglecting the remainder term in the interpolation.

This indicates the outstanding feature of the book, namely, the investigation of exact theorems with remainder terms. The work needed to be done, and it has been done extraordinarily well, the treatment being original, clear and elegant, and not really difficult considering the nature of the subject. His insistence on precise ascertainment of the limits of error in every case may at first be discouraging to the student, and may give him the impression that interpolation formulas are of no use unless he knows so much about the function as to render interpolation formulas unnecessary. But the author modifies the absoluteness of his language, and grants that in certain cases where it is not possible to give satisfactory proof that the remainder term is very small, it may yet be possible to assert that it is *probably* very small. He remarks also that in practical work cases occur where one is forced to interpolate even with differences of a high order, without having any idea whatever as to the influence of the remainder term, and that in such cases the result of an interpolation must be regarded as a more or less probable hypothesis and not as a mathematically proved fact.

In warning the student against the indiscriminate application of polynomials for interpolation purposes he discusses at some length the remarkable

case of the function $\frac{1}{1+x^2}$, a function which is everywhere continuous and everywhere possesses differential coefficients of every order; and he shows that if we take the interval between $x=-5$ and $x=+5$, and use equidistant values at equal intervals as the basis of the scheme of differences, interpolation gives worse and worse results the greater the number of values taken.

All the well-known interpolation formulas are discussed in detail, and the remainder term is derived for each. We think that it would have been a help to the student if the important principle, which is suggested and implied here and there, had been clearly and distinctly stated that in a given scheme of differences every formula which ends with a specified difference gives the same identical value of the function for a given value of the argument. It is very illuminating to the student to discover that each difference in a scheme of differences has several invariable associations. In interpolating by means of a given scheme of differences to obtain a particular value required, it is the fact that whatever formula be employed, if we make a point of ending with the same difference the same value is obtained, the error is the same, the remainder term is the same, the coefficient of the next term is the same whether it be a term of the formula or the remainder term, and in continuing the formula there is a choice of one or other of two differences and two only in calculating the next term.

The individuality of the formulas is emphasised so strongly that the student may be in danger of forming the erroneous impression that different formulas give different values although ending at the same difference. He should learn to use the scheme of differences with freedom and not to be bound by the precise form and order of the standard formulas. In a given scheme we can start with any value of the function and find our way to a given difference by any route, so that, for example, if we are working to a given fifth difference we can put down 16 formulas ending with that difference, and if we include formulas which stop short of that difference, there are 41 formulas available, the formulas being composed of terms each of which involves a single difference only. If we include terms which as in Sterling's or Bessel's formulas employ the mean values of two differences, and if we further include formulas like Everett's which run along a double line, then the number of available formulas is multiplied, but the principle we have laid down remains true if we end with a term which includes the given single difference only.

The remainder term in an interpolation formula may be verbally described as the next term with the substitution of a differential coefficient for a difference. The substituted differential coefficient, however, is for some unknown value of the argument within the limits which have entered into the scheme; and the use of the remainder term implies therefore knowledge of the limits within which the differential coefficients lie. In this connection he points out the value of what he calls the "Error Test."

The error test may be described as follows. Consider the value given by a formula which ends by the inclusion of a given difference; and consider the effect of including "the next term," *i.e.* the first of the excluded terms. The remainder term, or the error, of the formula is less than the next term, and has the same sign, if and only if the inclusion of the next term would alter the sign of the remainder term. The importance of the error test lies in the fact that we often have sufficient information to give us the sign of the remainder term, and the error test then allows us to use the "next term" in place of the remainder term.

But what is the next term? Following upon the inclusion of any difference in a formula, there is always (except at the beginning and end of a scheme) a choice between two alternatives for the next term, and we think it is of importance that the two alternatives should in every case be examined. We endeavour to illustrate this later with some practical examples.

We may remark here that one of the difficulties to students of the subject is the lack of convenient numerical examples for the illustration of danger points. The student is given examples of simple polynomials which convey to him the impression that if he will only go far enough he will obtain exact results. Or examples are taken from tables of trigonometrical functions

which are chosen because of their familiarity and because complete verification of the results is always possible. But these functions are free from singularities, and run with such perfect regularity that again the student is impressed by the fact that he invariably improves his results by including more terms. We want examples drawn from complex functions in which singularities are present in order to impress upon the student that in order to obtain the best results he will need all the knowledge and judgment he can bring to bear not only of the theory of the subject but of the facts represented by the function.

Professor Steffensen makes a virtue of necessity, and explains that his examples have not been chosen with a view to taxing the methods to the utmost, but rather with a view to avoiding heavy numerical work which would deter the student from going through examples by himself. We are sure that the serious student must not be afraid of heavy numerical work. In Table I. appended we give values of the function $l_x = \frac{1}{(1+\alpha)^{c^x}}$, an expression which is known to actuaries as the Gompertz Formula. By giving appropriate values to the constants this expression has been found to give with a remarkable approach to accuracy the number living in certain life tables at age x for values of x from the age of 30 upwards. For our present purpose we have fixed the value of c so that $c^8 = 2$, and of α so that $(1+\alpha)^8 = 1{\cdot}003$. These values are chosen for facility of computation and not to represent any life table in particular, but the resulting life table approximates to actual life tables sufficiently to allow us to infer that interpolation methods which are suitable for us in this arbitrary life table may be employed with confidence in most other life tables.

On account of the special values we have chosen for the constants, the intensity of mortality $\mu_x \equiv -\frac{1}{l_x}\frac{dl_x}{dx} = c^x \lambda c \lambda (1+\alpha)$, where λ is $\log_e$, doubles every eight years, and $l_{x+8} = l_x{}^2$ for all values of x. These properties facilitate the calculation of values of l_x, and these have been obtained by direct calculation for every value of x. In Table I. values of l_x, Dl_x and $D^2 l_x$ $\left(D = \frac{d}{dx}\right)$ are given for ages 60 to 96 to ten decimal places, l_x being a decimal fraction proportional to the number living and always less than unity.

In Table II. the values of $(10\Delta)^n$ are given for all powers to the 8th. The object of this table has been chiefly to show where the singular points fall, and to save space the differences are not given beyond the 4th decimal place. But differences of the 5th to 8th orders are given to the utmost figure allowed by Table I. and even beyond, the original calculations having been made to more than 12 decimal places. The first to fourth differences can easily be obtained to further decimal places by differencing the primary figures in Table I.

The differential coefficients of the function can be easily obtained to any order required. $Dl = -\mu l$, and since $D\mu = \mu \,.\, \lambda c$, it follows that

$$D^2 l = \mu l \,(\mu - \lambda c).$$

Writing D_c for $\frac{D}{\lambda c}$ and h for $\frac{\mu}{\lambda c}$, it is easily found that

$$D_c\,(D_c - 1)(D_c - 2) \ldots (D_c - n + 1)\, l = (-h)^n l.$$

The expression on the left is factorial in form, and by taking advantage of the table of *differences of nothing* used for expressing powers in terms of factorials (see page 55 of Professor Steffensen's volume) we can write down the following expressions:

$$D_c l = -hl, \quad D_c{}^2 l = -hl\,(1-h), \quad D_c{}^3 l = -hl\,(1 - 3h + h^2),$$
$$D_c{}^4 l = -hl\,(1 - 7h + 6h^2 - h^3),$$
$$D_c{}^5 l = -hl\,(1 - 15h + 25h^2 - 10h^3 + h^4),$$
$$D_c{}^6 l = -hl\,(1 - 31h + 90h^2 - 65h^3 + 15h^4 - h^5),$$

$\lambda c = {\cdot}0866 \ldots$ so that since $D_c = \frac{D}{\lambda c}$ and $h = \frac{\mu}{\lambda c}$,

$D^6 l = ({\cdot}0866 \ldots)^5 \mu l \times$ a polynomial in h,

and similarly for other powers of D. The values of the differential coefficients of any order can therefore be obtained without great difficulty if desired.

It will be noticed that $D^2 l_x = 0$ when $h = 1$. This happens when x is $= 75 +$ a small fraction.

We shall use the Table of l_x for the purpose of a few illustrative calculations.

I. *Bisection of the interval,* or as Professor Steffensen describes it, interpolation to halves. In Chapter VIII., dealing with the construction of tables, formulas are given for the subdivision of intervals according to the methods used by Briggs just 300 years ago in the construction of tables of logarithms by interpolation. We take the simple case of bisection only, and we use the Newton-Sheppard system of adjusted differences.

Let $\bar{A}_{2n} \cdot u_0$ denote $(u_n - u_{-n})/2n$ so that $\bar{A}_{2n} = (E^n - E^{-n})/2n$.

Then $\bar{A}_1^2 = E^1 - 2 + E^{-1}$ and $\bar{A}_2^2 = (E^2 - 2 + E^{-2})/4$, and it follows that

$$\bar{A}_2^2 = \bar{A}_1^2(1 + \tfrac{1}{2}\bar{A}_1^2),$$
$$\bar{A}_2^4 = \bar{A}_1^4(1 + \tfrac{1}{2}\bar{A}_1^2 + \tfrac{1}{16}\bar{A}_1^4),$$
$$\bar{A}_2^6 = \bar{A}_1^6(1 + \tfrac{3}{4}\bar{A}_1^2 + \tfrac{5}{16}\bar{A}^4 \ldots),$$
$$\bar{A}_2^8 = \bar{A}_1^8(1 + \bar{A}_1^2 + \ldots),$$
$$\bar{A}_2^{10} = \bar{A}_1^{10} \ldots,$$

and hence to 10th even differences :

$$\bar{A}_1^{10} = \bar{A}_2^{10},$$
$$\bar{A}_1^8 = \bar{A}_2^8 - \bar{A}_1^{10},$$
$$\bar{A}_1^6 = \bar{A}_2^6 - \tfrac{3}{4}\bar{A}_1^8 - \tfrac{5}{16}\bar{A}_1^{10},$$
$$\bar{A}_1^4 = \bar{A}_2^4 - \tfrac{1}{2}\bar{A}_1^6 - \tfrac{1}{16}\bar{A}^8,$$
$$\bar{A}_1^2 = \bar{A}_2^2 - \tfrac{1}{4}\bar{A}_1^8.$$

Thus if we start with adjusted differences formed from values of the function for arguments separated by intervals of two units it is an easy operation to obtain the corrected differences appropriate to intervals of a single unit. Then beginning with the highest differences we can interpolate for the intermediate values of the differences and finally of the function by using the formula

$$v_1 + v_{-1} - \bar{A}_1^2 v_0 = 2v_0.$$

Table A 1 gives, for the even values of x from 72 to 82, values of l to 8 figures and of the adjusted even central differences. Table A 2 gives the corrected differences and the interpolated values. In no case is there an error of more than a unit in the eighth figure.

We may remark in passing that there is no difficulty in adapting the process so as to give the successive even differential coefficients by means of the repetition of similar operations.

In Table A 2 we have bisected each interval separately, but it would have been sufficient to go through the operations for a single interval, and then after inserting the odd powers of the differences in that interval extend the scheme upwards and downwards by continuous addition so as to obtain interpolated values of the function bisecting three intervals on each side of the interval selected.

TABLE A 1. ADJUSTED DIFFERENCES.

x	l_x	$\bar{A}_2^2$	$\bar{A}_2^4$	$\bar{A}_2^6$	$\bar{A}_2^8$	$\bar{A}_2^{10}$
72	46447372	− 61482·00	+2532·687	− 87·015	+ ·347	+ ·358
74	40174254	− 23430·75	+2296·438	−120·829	+ 3·653	+ ·432
76	33807413	+ 23806·25	+1576·875	−140·031	+ 8·687	− ·152
78	27535797	+ 77350·75	+ 297·187	−124·484	+13·114	− ·789
80	21573584	+132084·00	−1480·437	− 56·485	+14·387	−2·012
82	16139707	+180895·50	−3484·000	+ 69·062	+ 7·613	−2·214

TABLE A 2. CORRECTED DIFFERENCES, AND INTERPOLATED VALUES OF THE DIFFERENCES AND OF THE FUNCTION.

x	l_x	Δ_1^2	Δ_1^4	Δ_1^6	Δ_1^8	Δ_1^{10}
72	46447372	− 62126·056	+2576·225	− 87·074	− ·011	+ ·358
	43332979·57	− 44333·148	+2520·015	−105·904	+ 1·408	+ ·395
74	40174254	− 24020·225	+2357·900	−123·326	+ 3·221	+ ·432
	36991508·28	− 1349·567	+2072·749	−137·959	+ 5·960	+ ·140
76	33807413	+ 23393·840	+1649·639	−146·631	+ 8·839	− ·152
	30646711·59	+ 49786·823	+1079·920	−146·501	+11·607	− ·471
78	27535797	+ 77259·725	+ 363·700	−134·763	+13·903	− ·789
	24502142·36	+105096·282	− 487·023	−109·512	+15·852	−1·401
80	21573584	+132445·815	−1447·258	− 68·408	+16·399	−2·012
	18777471·36	+158348·289	−2476·345	− 10·236	+14·170	−2·113
82	16139707	+181774·417	−3515·668	+ 62·107	+ 9·827	−2·214

In Table B 1 a scheme is given of terms for building up formulas for the calculation of l_{75} being given neighbouring values of l to eight figures for even arguments. The initial term, which is always a value of l, is not given in the scheme, but can be obtained from Table I. Starting with any value of l and going always to the right, we have the choice at each step of taking the next term above to the right or the next term below to the right. Thus starting with l_{72} the terms of a formula ending with a 6th difference are as follows :

+ve terms	−ve terms
46,447,372	
	9 409 677
	35 146·1
	11 809·3
861·2	
	134·9
46,448,271·0	9,456,767·3
9,456,767·3	
36,991,503·7	

Starting with l_{76} we give the terms of another formula ending with the same 6th central difference :

+ve terms	−ve terms
33,807,413	
3,135,808	
35,709·4	
809·3	
861·2	
	44·3
	52·8
36,991,601·9	97·1
97·1	
36,991,503·8	

Each formula ends with the inclusion of the same difference, and gives the same approximate value, namely, 36,991,504, which differs from the true value by a deficiency of 4 in the last figure. The next term bringing in a 7th difference would leave an error of a unit, and this error would be removed by going to 8th differences.

TABLE B 1. TABLE OF TERMS OF FORMULAS.

x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8
	−9,040,785							
			+34,900·6		−48·9		−9·9	
				+11,080·5		+114·3		+·3
72		−92,223		−1,582·9		−38·1		−·1
	−9,409,677		+47,563·6		+103·3		−9·5	
	−3,136,559		−9,512·8		−44·3		+5·3	
		−35,146·1		−1,435·3		−52·8		−1·3
74		+11,715·4		+861·2		+37·8		+1·0
	−3,183,420·5		−11,809·3		−134·9		+3·0	
	+3,183,420·5		+11,809·3		+134·9		−3·0	
		−11,903·1		+591·3		+43·8		+2·4
76		+35,709·4		−985·6		−61·3		−3·1
	+3,135,808		+13,386·1		+239·9		+2·4	
	+9,407,424		−66,930·6		−559·9		−4·4	
				−185·7		−54·5		−4·6
78		+116,026·1		+1,300·2		+163·5		+10·0
	+8,943,319·5		−68,416·6		−777·7		−19·1	

Each term in Table B 1 consists of a coefficient multiplied by a difference, and it may be convenient to put the coefficients on record in Table B 2, since they can be employed for any case of bisection. The coefficients are calculated for use with adjusted differences, and themselves constitute a scheme of adjusted differences reversed and inverted.

TABLE B 2.

	Coefficient following							
x	l	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
72	3		$\frac{5}{2}$		$-\frac{7}{8}$		$\frac{9}{16}$	
		$\frac{3}{2}$		$-\frac{5}{8}$		$\frac{7}{16}$		$-\frac{45}{128}$
74	1		$-\frac{1}{2}$		$\frac{3}{8}$		$-\frac{5}{16}$	
		$-\frac{1}{2}$		$\frac{3}{8}$		$-\frac{5}{16}$		$\frac{35}{128}$
76	−1		$\frac{1}{2}$		$-\frac{3}{8}$		$\frac{5}{16}$	
		$\frac{3}{2}$		$-\frac{5}{8}$		$\frac{7}{16}$		$-\frac{45}{128}$
78	−3		$-\frac{5}{2}$		$\frac{7}{8}$		$-\frac{9}{16}$	

In Table B 3 we give a table of errors showing the error when any term in Table B 1 is made the final term of the formula. In twenty-one cases we have the opportunity of comparing the error with the two alternatives which are available in Table B 1 for the next term. In three cases the alternatives have opposite signs. This is, of course, a danger signal. In such a case we must either stop short and use no more terms, or else we must include two more orders of differences at least. It may be noted that when the alternatives for the next term have opposite signs the value has the least error for its own order of differences. In eighteen cases the alternatives for the next term have the same sign, and in sixteen out of the eighteen the error lies between the two alternatives. In the two cases where the error does not lie between them they approximate closely to one another and to the error in value.

III. *Osculatory Interpolation.* The examples given above show that if values of l_x are taken at intervals of two years it is necessary to use differences of the eighth order to obtain interpolated values correct to eight significant figures. An example will now be given to show the great increase of power which is gained by the introduction of Differential Coefficients into the scheme of differences. Dr. Thomas Bond Sprague, who first used the system many

TABLE B 3.

	Error involved in stopping at the term which includes :							
	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8
	−415,078·5		11,135·9		104·2		−·1	
72		46,036·5		55·3		−10		−·4
	−46,186·5		−1,527·6		−48		−·5	
74		−11,040·4		−92·3		4·8		·8
	6·75		768·9		42·6		1·9	
76		12,578·1		177·6		−·9		−·3
	48,287·5		−808		−62·3		−3·4	
78		−67,738·6		−622·3		−7·9		1·2
	512,392		678		155·4		11·2	

years ago, gave it the name of Osculatory Interpolation. In the present volume it is called Osculating Interpolation, and notes on the subject will be found in Chapter IV. It should be mentioned however that Dr. Sprague did not show how osculatory interpolation might be regarded as a special case of ordinary Newtonian Interpolation, the credit for which must be given to Thiele. It will be readily appreciated that the introduction of Differential Coefficients springs quite naturally from the use of Newton's divided differences, or from the Newton-Sheppard system of adjusted differences.

In Table C two values l_{64} and l_{68} are taken with an interval of four years between them, and the scheme of interpolation is based upon these values and their first and second differential coefficients. The system of adjusted differences is employed, and an adjusting factor is shown before each adjusted difference. If v_0 and v_1 be any two successive values or differences, $\Delta(01)$ their adjusted difference and $\alpha(01)$ the adjusting factor for that difference, then $v_1 = v_0 + \alpha(01) \times \Delta(01)$. The scheme of differences is first completed down to the figures underlined, and is then extended on the assumption that 5th differences are constant, so that interpolated values of l_{65}, l_{66}, and l_{67} are obtained by a continuous process. The results differ from the true values by not more than a unit in the 8th figure.

TABLE C. EXAMPLE OF OSCULATORY INTERPOLATION.

x	l_x	Adjusting Factor	Adjusted Differences of 1st Order	Adjusting Factor	Adjusted Differences of 2nd Order	Adjusting Factor	Adj. Diff. of 3rd Order	Adjusting Factor	Adj. Diff. of 4th Order	Adjusting Factor	Adj. Diff. of 5th Order
68	58144142										
		$(\frac{3}{5})$	−02731725								
68	58144142			$(\frac{3}{5})$	−00108344						
		$(\frac{3}{5})$	−02731725			$-\frac{1}{5}$	$0^4 4873$				
68	58144142			$-\frac{1}{5}$	−00114842			$-\frac{1}{5}$	$0^4 1778$		
	−4		−02502042			$-\frac{1}{5}$	$0^4 3095$			$-\frac{1}{5}$	$0^5 215$
64	68152309			$-\frac{1}{5}$	−00118968			$-\frac{1}{5}$	$0^4 1606$		
		$(\frac{3}{5})$	−02264105			$-\frac{1}{5}$	$0^4 1489$			$-\frac{2}{5}$	$0^5 215$
64	68152309			$(\frac{3}{5})$	−00120953			$-\frac{2}{5}$	$0^4 1477$		
		$(\frac{3}{5})$	−02264105			$\frac{1}{5}$	$0^4 0381$			$-\frac{2}{5}$	$0^5 215$
64	68152309			$\frac{1}{5}$	−00120826			$\frac{2}{5}$	$0^4 1391$		
		1	−02324518			$\frac{2}{5}$	$0^4 1076$			$\frac{2}{5}$	$0^5 215$
65	65827791			1	−00120109			$\frac{3}{5}$	$0^4 1520$		
		1	−02444627			$\frac{3}{5}$	$0^4 2216$				
66	63383164			1	−00117893						
		1	−02562520								
67	60820644										

We attach a good deal of importance to this result. It has been customary in the compilation of Life Tables to tabulate l_x, Δl_x, and μ_x. Since $Dl_x = -\mu_x l_x$ there would be no difficulty in tabulating Dl_x. When l_x can be expressed by a Gompertz expression or by the related expression due to Makeham, $D^2 l_x$ takes a very simple form and is easily tabulated. In other cases it is only necessary to settle the values of $D\mu_x$ and then $D^2 l_x$ is easily deduced. When great precision of calculation is desired, and for the investigation of many questions arising out of the use of life tables increased precision is much needed, we suggest that the systematic tabulation of Dl_x and $D^2 l_x$ would be a valuable advance on previous practice.

We are glad to observe that Professor Steffensen's book has already begun to make its influence felt in the University of Cambridge, and has suggested and stimulated the publication at the University Press of a paper by Messrs. R. A. Fisher, Sc.D., and J. Wishart "On the distribution of the error of an interpolated value, and on the construction of Tables."

TABLE I.

x	l_x		Dl_x		$D^2 l_x$	
60	·76252	30639	−·01791	24114	−·00113	12121
61	·74403	94599	−·01906	01270	−·00116	31691
62	·72439	32771	−·02023	63873	−·00118	80330
63	·70355	97931	−·02143	32650	−·00120	41073
64	·68152	30885	−·02264	10541	−·00120	95336
65	·65827	79040	−·02384	81172	−·00120	23114
66	·63383	16375	−·02504	07586	−·00118	03323
67	·60820	64492	−·02620	31401	−·00114	14286
68	·58144	14230	−·02731	72537	−·00108	34417
69	·55359	47179	−·02836	29732	−·00100	43106
70	·52474	56199	−·02931	82058	−·00090	21835
71	·49499	63825	−·03015	91669	−·00077	55533
72	·46447	37202	−·03086	08023	−·00062	34154
73	·43332	97989	−·03139	73772	−·00044	54447
74	·40174	25447	−·03174	32501	−·00024	21847
75	·36991	50849	−·03187	38376	−·00001	52394
76	·33807	41283	−·03176	67657	+·00023	25489
77	·30646	71116	−·03140	31843	+·00049	69544
78	·27535	79656	−·03076	92001	+·00077	22807
79	·24502	14187	−·02985	73570	+·00105	13585
80	·21573	58368	−·02866	80633	+·00132	56581
81	·18777	47146	−·02721	08383	+·00158	55417
82	·16139	70723	−·02550	52281	+·00182	06761
83	·13683	71700	−·02358	12267	+·00202	06085
84	·11429	41163	−·02147	90433	+·00217	54923
85	·09392	20905	−·01924	80864	+·00227	69203
86	·07582	20092	−·01694	50887	+·00231	87946
87	·06003	54956	−·01463	13840	+·00229	81343
88	·04654	19513	−·01236	94572	+·00221	56996
89	·03525	93434	−·01021	90148	+·00207	63087
90	·02604	90149	−·00823	29383	+·00188	87369
91	·01872	44111	−·00645	35767	+·00166	51372
92	·01306	31450	−·00490	98565	+·00141	99905
93	·00882	13591	−·00361	56410	+·00116	86839
94	·00574	89771	−·00256	96213	+·00092	59034
95	·00360	42607	−·00175	68048	+·00070	40941
96	·00216	61532	−·00115	13974	+·00051	22529

TABLE II.

x	10Δ	$10^2\Delta^2$	$10^3\Delta^3$	$10^4\Delta^4$	$10^5\Delta^5$	$10^6\Delta^6$	$10^7\Delta^7$	$10^8\Delta^8$
60		−·1131		+·0558		+·1739		−·6008
	−·1848		−·0318		+·1524		−·0225	
61		−·1163		+·0711		+·1716		−·8037
	−·1965		−·0247		+·1696		−·1029	
62		−·1187		+·0880		+·1614		−1·0409
	−·2083		−·0159		+·1857		−·2069	
63		−·1203		+·1066		+·1407		−1·3041
	−·2204		−·0053		+·1998		−·3374	
64		−·1208		+·1266		+·1069		−1·5744
	−·2325		+·0074		+·2104		−·4948	
65		−·1201		+·1476		+·0574		−1·8555
	−·2445		+·0222		+·2162		−·6803	
66		−·1179		+·1692		−·0106		−2·0942
	−·2563		+·0391		+·2151		−·8898	
67		−·1140		+·1908		−·0996		−2·2775
	−·2677		+·0582		+·2052		−1·1175	
68		−·1082		+·2113		−·2113		−2·3442
	−·2785		+·0793		+·1840		−1·3519	
69		−·1002		+·2297		−·3465		−2·2361
	−·2885		+·1023		+·1494		−1·5755	
70		−·0900		+·2446		−·5041		−1·8854
	−·2975		+·1267		+·0990		−1·7641	
71		−·0773		+·2545		−·6805		−1·2186
	−·3052		+·1522		+·0309		−1·8859	
72		−·0621		+·2576		−·8691		−·1512
	−·3114		+·1779		−·0560		−1·9011	
73		−·0443		+·2520		−1·0592		+1·3637
	−·3159		+·2031		−·1619		−1·7647	
74		−·0240		+·2358		−1·2356		+3·3499
	−·3183		+·2267		−·2855		−1·4297	
75		−·0013		+·2073		−1·3786		+5·7923
	−·3184		+·2474		−·4233		−·8505	
76		+·0234		+·1649		−1·4637		+8·5463
	−·3161		+·2639		−·5697		+·0042	
77		+·0498		+·1080		−1·4632		+11·3998
	−·3111		+·2747		−·7160		+1·1441	
78		+·0773		+·0364		−1·3488		+13·9988
	−·3034		+·2784		−·8509		+2·5440	
79		+·1051		−·0487		−1·0944		+15·8032
	−·2929		+·2735		−·9603		+4·1243	
80		+·1324		−·1447		−·6820		+16·2733
	−·2796		+·2590		−1·0285		+5·7517	
81		+·1583		−·2476		−·1068		+14·6863
	−·2638		+·2343		−1·0392		+7·2203	
82		+·1818		−·3515		+·6152		+10·5329
	−·2456		+·1991		−·9777		+8·2736	
83		+·2017		−·4493		+1·4426		+3·4337
	−·2254		+·1542		−·8334		+8·6170	
84		+·2171		−·5326		+2·3043		−6·4312
	−·2037		+·1009		−·6030		+7·9738	
85		+·2272		−·5929		+3·1016		−18·3360
	−·1810		+·0416		−·2928		+6·1402	
86		+·2314		−·6222		+3·7157		−30·6998
	−·1579		−·0206		+·0787		+3·0703	
87		+·2293		−·6143		+4·0227		−41·2660
	−·1349		−·0820		+·4810		−1·0563	
88		+·2211		−·5662		+3·9171		−47·3278
	−·1128		−·1387		+·8727		−5·7891	
89		+·2072		−·4790		+3·3381		−46·2734
	−·0921		−·1866		+1·2065		−10·4165	
90		+·1886		−·3583		+2·2965		−36·4514
	−·0732		−·2224		+1·4362		−14·0616	
91		+·1663		−·2147		+·8903		−17·9526
	−·0566		−·2439		+1·5252		−15·8569	
92		+·1419		−·0622		−·6953		+6·7869
	−·0424		−·2501		+1·4557		−15·1782	
93		+·1169		+·0834		−2·2132		
	−·0307		−·2417		+1·2343			
94		+·0928		+·2068				
	−·0214		−·2211					
95		+·0707						
	−·0144							
96								

D. C. FRASER.

Integral Bases. By W. E. H. BERWICK, Sc.D. Pp. 95. 6s. 6d. 1927. (Cambridge University Press.)

Many important advances in mathematics had their origin in the study of indeterminate equations. This is not surprising on considering the ease with which questions of this type suggest themselves to people even with little mathematical training. One of the most important theorems is Fermat's Last Theorem, namely, that if n is a given integer greater than two, the equation

$$x^n - y^n = z^n$$

has no integer solutions in x, y or z unless one of them is zero. Many efforts to prove this proceed by factorising the left-hand side as

$$(x-y)(x-\zeta y) \dots (x-\zeta^{n-1}y) = z^n,$$

where $\zeta = e^{2\pi i/n}$ is an nth root of unity.

To draw any conclusion from this, one considers an elementary analogy. Suppose we are given

$$a_1 a_2 \dots a_n = z^n,$$

where $a_1, a_2, \dots a_n, z$ are integers. Then if no two of $a_1, a_2, \dots a_n$ have a common factor, it follows that

$$a_1 = \pm x_1^n, \quad a_2 = \pm x_2^n, \dots,$$

where $x_1, x_2, \dots$, are integers. Can a similar conclusion be drawn about $x-y, x-\zeta y, \dots$, and are they nth powers of similar expressions? An answer to this question requires a closer study of these complex numbers and in particular of their arithmetical properties, which it is conceivable may be entirely different from those holding for the ordinary integers. Similar questions occur for other types of irrationalities than a root of unity. The whole subject is now included in what is known as the Theory of Algebraic Numbers.

A number x is called an algebraic number if it is a root of an irreducible equation of the form

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0,$$

where $a_0, a_1 \dots a_n$ are integers of which the first is not zero. Any rational function of x, say y, with rational coefficients, can be expressed in the form

$$y = A_1 x^{n-1} + A_2 x^{n-2} + \dots A_n,$$

where $A_1, A_2, \dots A_n$ are rational numbers. The assemblage of all these rational functions of x is called the field $R(x)$.

Amongst algebraic numbers, the most important are those called algebraic integers corresponding to the case when $a_0 = 1$. We put then $x = \theta$ say, and must now distinguish between algebraic integers and the ordinary integers which we call the rational integers. The properties of the former have some resemblance to those of the latter. Thus the sum, difference or product of two algebraic integers is also an algebraic integer; but in general they cannot be factorised in a unique manner by means of integers in the field $R(\theta)$, *e.g.* in $R(\sqrt{-5})$,

$$21 = 3 \times 7 = (4+\sqrt{-5})(4-\sqrt{-5}) = (1+2\sqrt{-5})(1-2\sqrt{-5}),$$

and none of these factors of 21 can be expressed as the product of integers of the field $R(\sqrt{-5})$. The field $R(x)$ includes an infinity of algebraic integers, certainly those in which $A_1, A_2 \dots$ are all rational integers, though it is not necessary that $A_1, A_2 \dots$ should be integers. This is shown, for example, when

$$\theta^2 = 5, \quad y = \frac{1+\theta}{2},$$

where y is an algebraic integer since

$$y^2 - y = 1.$$

It is obviously an important question to find a simple expression for the algebraic integers in a field. It can be shown that if $\omega_1, \omega_2 \dots \omega_n$ are suitably chosen algebraic integers in the field $R(\theta)$, then any other integer I in this field can be expressed uniquely in the form

$$I = m_1\omega_1 + m_2\omega_2 + \dots m_n\omega_n,$$

where m_1, m_2, ... are rational integers. The numbers ω_1, $\omega_2 \ldots \omega_n$ are called the base or integral base of the field. Each of them can be expressed in the form

$$D\omega_1 = \omega_{11} + \omega_{12}\theta + \ldots \omega_{1,\,n-1}\theta^{n-1},$$

where ω_{11}, ω_{12}, etc., are rational integers and D is the discriminant of the equation in θ. There is no theoretical difficulty in finding them in any particular case. General expressions have also been given when θ is the root of a quadratic or cubic equation.

Professor Berwick's main object in this tract is the question of finding the integral base for the field $R(\theta)$ defined by $\theta^n = a$. He develops, however, methods applicable to the more general algebraic number. No reference is made to the important work during recent years of Ore, who has anticipated the author in his general methods and results. Thus let p be any prime number, then the method depends upon finding the integers which are of the form

$$\frac{a_{20} + a_{21}\theta}{p}, \quad \frac{a_{30} + a_{31}\theta + a_{32}\theta^2}{p^2}, \text{ etc.},$$

where the a's are rational integers. The integral base is easily found if this process is carried out for all the prime factors of the discriminant D. A geometrical method of representation is used similar to that of Newton's polygon for discussing the forms of a plane curve near a multiple point. The method developed is not exhaustive, and there are special cases which require further discussion, or additional stages. It is, however, sufficient for his object. The discussion of the field $R(\sqrt[n]{a})$ requires the consideration of as many as twenty-three cases depending upon the common factors of a and n, their parity, etc., and all of them are dealt with. These results are an important contribution to the theory of algebraic numbers, and will prove a very useful source of material in any arithmetical investigation dealing with them.

The tract is more for the experienced mathematician than for the beginner. The generality of the methods and the details to be borne in mind, both combine to make the book one which requires considerable effort from the reader. Perhaps a mathematical journal might have been a more natural place for publication, but then the results would not have been so convenient for reference. In any case, it is very enterprising of the University Press to include it as one of the Cambridge tracts in mathematics. L. J. MORDELL.

A Course of Modern Analysis. By E. T. WHITTAKER and G. N. WATSON. Fourth Edition. Pp. vi+608. 40s. net. 1928. (Camb. Univ. Press.)

The fourth edition of this standard treatise does not differ much from the third. All the misprints with which the writer was previously acquainted appear to have been corrected; and the list of references is more complete. There are no new chapters, or discussions of new types of special functions. The general excellence of the work, which stands quite alone, is so well recognised everywhere that a new edition, in these circumstances, leaves little for a reviewer to say.

But we still have certain regrets, from the point of view of the applied mathematician, regarding the fundamental introduction of the special functions as contour integrals rather than from differential equations. This, however, is no hardship to the mathematical analyst, for to him it gives a comprehensive outlook on the whole class of functions as special cases of a very general theory of analysis, which it is the fundamental aim of the book to expound. This is the reason why, as the authors say, the Jacobian elliptic functions, for example, are theoretically best visualised as quotients of Theta-functions—but not, for instance, by a student of higher dynamics, or allied subjects, who wishes to express his solutions in a canonical shape in terms of the Jacobian functions. As a teacher of mathematics, the writer has been driven to conclude that such students frequently cannot bear to "learn" the Theta-function theory as an approach to the functions sn, cn, dn, which are quite simple generalisations of trigonometric functions, and subject to tabulation.

Perhaps, in view of the aim of the work, this concession should not be made, but it was done in the original Whittaker's "Modern Analysis" preceding these four editions, and the change is a trouble to many students.

Spherical Harmonics: An Elementary Treatise on Harmonic Functions and their Applications. By T. M. MacRobert. Pp. xii+302. 15s. net. (Methuen.)

This treatise, as its name implies, is not limited to Spherical Harmonics, but contains considerable sections on Fourier Series and Bessel Functions, with the applications of each. It is clearly written primarily with a view to the need of the applied mathematician, or mathematical physicist—if there is any great distinction between these two classes of readers—for the applications mainly discussed,—potential functions, vibrations of strings, conduction of heat, and electrostatics,—belong equally to either. There is considerable variety in the applications, and the book should provide a solid guide, to an intelligent reader, as to how he should set about the solution of a new problem in any of these branches. For this reason alone, the book deserves a cordial welcome, as, in this respect, it does something which no other work has done with the definitely limited compass of analytical knowledge available.

For example, contour integration is avoided, and many quite competent students of physics have, in the experience of a teacher, had no opportunity to acquire a knowledge of this subject when they must begin the problems discussed. There is no avoidance, however, of imaginary quantities as such,—as the author puts it, the strict methods of the Real Variable theory are not adhered to. This is a further advantage, in that proofs of theorems are often extremely difficult when the variables are restricted throughout to real values.

The form of the book closely resembles that of Gray, Mathews, and MacRobert's Bessel functions, as does the arrangement of the material. Though some degree of rigour is maintained throughout, the book will necessarily make less appeal to the strictly pure mathematician, in that it avoids leading the reader, who wants the applications, through mazes of inequalities which are often needed for a thoroughly strict proof. We consider that the author has taken the proper middle course. We regret that the Legendre functions of imaginary argument, or spheroidal harmonics, are not discussed, though Bessel Functions of imaginary argument are included.

The work is a very desirable addition to any mathematical library.

W. O.

Lehrbuch der Ballistik, Vol. III.; Experimentelle Ballistik. By C. Cranz. 2nd edn. Pp. xii+408. 39 RM. 1927. (Springer, Berlin.)

This book is the third and concluding volume of Dr. Cranz's *Lehrbuch*. The first edition, published in 1913, gave an exhaustive account of experimental methods in use before the war; the present edition brings this account up to date, introducing many refinements to existing methods as well as a clear and concise description of new experimental processes.

Among the notable additions in the second edition are: various applications of thermo-electric methods of measurement; new methods and instruments for the determination of wind-velocity; measurement of muzzle velocity by means of a concussion target instead of a contact target; an electrostatic spark chronograph capable of measuring a time-interval of 0·0013 second with a probable error of 0·44 per cent.; a critical survey of the theory of the crusher gauge and of methods of calibrating it; new methods of measuring powder pressure; perfection of Mach's method of photographing a bullet in flight; new methods of ballistic kinematography; a method of taking photographs at the rate of 100,000 per second by means of a quenched spark; the use of an unstable, aperiodic, wireless-valve circuit for the measurement of velocity. An additional chapter is contributed by Dr. K. Becker on direct and indirect laying.

Many of the methods described are due to the author himself, and many others have been actually tried by him in his ballistic laboratory; this practical experience has enabled him to give an accurate as well as a practical account of the methods described, while his vast theoretical knowledge has ensured constant touch with ballistic theory.

The book concludes with a comprehensive bibliography and a large number of excellent photographs, most of which were taken with the author's rapid kinematograph.

There is one notable omission; no mention is made of the wind-channel work at high and low velocity which has been carried out in various countries since the war. This work has already added to our knowledge of air-resistance, and it opens a wide field for experimental ballistic research in the future.

The *Lehrbuch* as a whole is a most valuable contribution to the literature of the subject. F. R. W. HUNT.

The Principles of Thermodynamics. By GEORGE BIRTWISTLE. 2nd edn. Pp. ix + 168. 7s. 6d. net. 1927. (Cambridge University Press.)

In this new edition following the first after only two years Mr. Birtwistle, besides correcting one or two errors and adding a paragraph with extra references to the chapter on radiation, has added a final chapter of five pages on Nernst's heat theorem. I agree generally with the remarks made by the reviewer of the first edition of this book (who reviewed it in the *Gazette* for January 1926), and would only add that some parts of it seem to me to have been compressed rather too much. I think it would be improved if, for example, the section :—

"6. *Perfect gas.* This is an ideal gas, infinitely far from the point of liquefaction, of which one of the properties is that of satisfying Boyle's and Charles' laws exactly, so that $pv = Rt$."—

were somewhat expanded; provided that this could be done without increasing the price of the book. L. H. THOMAS.

Das Rechnen der Naturvölker. By E. FETTWEIS. 5 M. 1927. (Teubner.)

In this volume, one of the first to be issued by Teubner since the war, the author gives an interesting and well-balanced account of the methods of counting adopted by primitive peoples in various parts of the globe. The general run of its contents can best be illustrated by short extracts.

(1) "*I was not a little astonished to find that the Zunis did not consider the two hands held up apart as meaning ten. I can illustrate this and its cause by my first experience of it as a Zuni. Whilst I was serving an apprenticeship with the chief silversmith of Zuni, in* 1881, *he asked me one day how many silver dimes I had. As my mouth was full of buttons I held up both hands spread out and apart, to assure him that I had ten ten-cent pieces. "Alas, son!" said he, "I already have two half-dollars, but I was hoping you had ten-cent pieces enough to exchange for them." "But I have," said I, ejecting the buttons and resuming speech in my surprise; whereupon he laughed at my having "split the sign for ten" making it "two fives."*

(2) "*I once had to sit patiently for over half-an-hour while an old chief gave the names and counted over the fifteen men lost by his tribe in a head-hunting raid. Slowly he doubled down one finger for each name, first on one hand and then, keeping that first closed, doubled down the fingers of the other. Then he began on his toes and when at about twelve or thirteen found he had unwittingly opened his fingers of the right hand and, despite my protestations, started all afresh, persisting that the count was spoilt and that having opened his hand I should never be able to understand it unless repeated.*"

To those who can read German and enjoy this sort of entertainment, Herr Fettweis' book will provide many hours of attraction. It contains a useful bibliography on the subject. W. E. H. B.

Questioni riguardanti le matematiche elementari raccolte e coordinate de F. Enriques. Parte terza. **Numeri primi e analisi indeterminata. Massimi e minimi.** Pp. 471. 70 lire. 1927. (Zanichelli, Bologna).

This volume contains four separate papers on analytical subjects. The first, by the late U. Scarpis, deals with some miscellaneous parts of the theory of numbers under the headings: prime numbers, indeterminate equations, and congruences. In the next two, A. Padoa and O. Chisini give algebraic and geometrical examples of maxima and minima from an elementary and historical standpoint, without going into the existence theorems treated by F. Enriques in the last, which also includes some calculus of variations.

All four are written so as to be intelligible and interesting without specialist knowledge of the particular subjects; the last has perhaps suffered somewhat from this restriction, especially where it touches on the theory of aggregates.

H. P. H.

Die Scherpe Hoek. By P. J. Van den Berg. Pt. I., pp. 22; Pt. II., pp. 25; Pt. III., pp. 22. n.p. 1928.

A curious work devoted to the evaluation of π which the author seems to claim the power of expressing exactly in rational numbers. He obtains two interesting series of angles by working on properties of tangents only. He uses inverse notation but avoids the symbols $\tan^{-1}x$, arc $\tan x$ by means of the symbol $\angle x$ as a substitute, though he is not consistent in this, as we find $\angle 15°$, $\angle 22° 30'$, $\angle 30°$, etc., in which the sign has its ordinary meaning. He writes the series in question for $\frac{\pi}{8}$, $\frac{\pi}{4}$ thus:

$$\text{(i)}\quad \angle\frac{1}{3}+\angle\frac{1}{17}+\angle\frac{1}{99}+\angle\frac{1}{577}\cdots;$$

$$\text{(ii)}\quad \angle\frac{1}{\sqrt{2}}+\angle\frac{1}{5\sqrt{2}}+\angle\frac{1}{29\sqrt{2}}+\angle\frac{1}{169\sqrt{2}}\cdots;$$

for results which may be conveniently given thus:

$$\frac{\pi}{8}=\sum_1^\infty \tan^{-1}\frac{1}{p_{2r}},\qquad \frac{\pi}{4}=\tan^{-1}\frac{1}{q_{2r-1}\sqrt{2}},$$

where

$$\frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \frac{q_3}{q_3}\cdots$$

are the successive convergents to the continued fraction for $\sqrt{2}$.

From these and other uses of the tangent formulae the writer obtains many equations, and may justifiably claim credit for great ingenuity and industry. We have worked carefully through several pages to check his calculations, and so far we have found them correct, but we have neither the time nor the courage to attack the thirty or so more, and we are the less inclined to do so as we do not see how the final value obtained can be true to a greater number of significant figures than the numerical results on p. 8 of Pt. II. which appear to be merely tabular. Though we have not detected any serious error in computation, we have noticed several slips which may be due to accidental omission or to errors in transcribing and printing and seem to imply want of due correction for the press. Care in this respect might have helped to remove some obscurity from the two introductory allegories with which the work opens. This may be intentional in the account of the mystic casket, but scarcely so in the instructions given by the pretty Angelina to the smart young surveyor for evaluating the curious expression $1\frac{985144329}{1106913711}\sqrt{3}$ which he had found on a scrap of paper.

E. M. Langley.

History of the Sciences in Greco-Roman Antiquity. By A. Raymond. Translated by Ruth Gheury de Bray. Pp. x+239. 7s. 6d. net. 1927. (Methuen.)

This volume can hardly be called a history of the science, as the greater part of it deals with mathematics and the sciences into which they have "intruded." The space denoted to biology and medicine, for example, is so inadequate as to make the reader wonder why these subjects were treated at all.

Looking upon the book as an elementary history of Greek mathematics and of the influence of the Greek mathematicians upon the development of astronomy, mechanics and geography, it may be said that it gives a very readable sketch of the wonderful story of Greek achievement in these fields. The author does not seem to be fully conversant with what is known and what is conjectured as to the indebtedness of the Greeks to their predecessors, nor is he always entirely trustworthy in the dates, having yet to discover the virtues of an "about."

The translator is to be congratulated on her share in the production of the volume.

The Byzantine Astrolabe at Brescia. By O. M. DALTON. Pp. 14. 2s. net. 1927. (Milford, Oxford Univ. Press.)

This reprint from the *Proceedings of the British Academy* gives a scholarly account of a notable astrolabe in the Museo dell' Età Cristiana at Brescia. The three excellent photographs are of the front, showing the *arachne* or *rete*, with inscriptions; the "Mother" (*the Moder of thyn Astrolabie is the thikkeste plate* . . . says Chaucer), engraved as a "table"; and of the back, giving the date A.D. 1062.

The pamphlet opens with a general enumeration of the parts of an astrolabe, followed by a description of the Byzantine example. Four iambic verses engraved on the *arachne* tell us that "Sergius, the Persian, of consular rank, fashioned with ardent mind this intricate work." The rubbing of the date, here reproduced, runs: "Decree and command of Sergius, Protospatharios, consul and man of science (ΕΠΙCΤΗΝΟC = ? ἐπιστήμονος), in the month of July; fifteenth indiction, year 6570."

Thus, as the author says, "this instrument, inscribed and precisely dated, takes its place as an historical document; it has all the evidential value of a manuscript."

The conclusion is that "Byzantine astronomical science . . . was in evidence under the Macedonian and Comnenian emperors. . . . Hellenistic science was remembered as late as A.D. 1062." The Brescian astrolabe "makes it probable that astronomical studies continued until the time of the Fourth Crusade."

Questions d'Arithmétique. By B. NIEWENGLOWSKI. Pp. viii + 225. 15 fr. 1927. (Vuibert, Paris.)

This collection of questions, few of which demand any serious mathematical equipment, will take any intelligent student safely over the threshold of the elementary Theory of Numbers. Some are of the type "plaisants et délectables" associated with the names of writers from Bachet down to Rouse Ball and Lucas. The opening chapter—"Curiosités"—is followed by one of 42 pages on the integral solutions of equations. The examples, graduated in difficulty from $ax \pm by = c$ to $x^2 - x + 1 = M(p)$, p being a prime, include the well-known problem of the three husbands and their three wives who go a-shopping, "according to plan" as to prices and quantities: how to pair off husband and wife. The problem, with French names, is here taken from a *Miscellany of Mathematical Problems* (1743), but I think it will be found, with Dutch names, in the first Raphson-Cunn edition of Newton's *Universal Arithmetick* at least some fifteen years earlier. It was set on June 8, 1877, "at the University of Cambridge," and was thereupon communicated by Mansion to the *Nouvelle Correspondance* (1878, p. 145). M. Niewenglowski quotes a protest from Catalan on the subject of such questions: "Voici un exemple, assez grossier, de l'art d'ensevelir le fond sous les accessoires. Si l'arithméticien qui, en 1743, inventait cette question ridicule; si l'Université de Cambridge, qui a cru devoir la tirer d'un juste oubli, n'ont pas dit, tout simplement: 'Résoudre en nombres entiers les équations

$$x^2 - x'^2 = z^2 - y'^2 = z^2 - z'^2 = 63, \quad x - y' = 23, \quad y - z' = 11,$$

c'est, sans doute, par respect pour la maxime: *plus on est obscur, plus on est savant*. Comme s'il y avait rien de préférable à la clarté et à la simplicité'!"

Our author dryly adds: "On trouvera sans doute E. Catalan bien sévère."

By easy stages the student is taken through sections applying easy indeterminate analysis to questions arising out of divisibility, periodic decimals, progressions, etc., to the sections treating of the applications of Fermat's and Wilson's Theorems, and finally to a chapter on the Pellian Equation.

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605. Our college lectures in algebra and logic were odious to me beyond description. . . . As to the elements of geometry and algebra, these are in themselves so extremely plain, so accessible to every capacity, and carry with them such beautiful and engaging evidence—truth in her very essence—that I can scarcely account for an indisposition to such theories but from a defect of judgment or dexterity in the teacher.—*Gilbert Wakefield* [at Jesus, Cambridge 1772], P.C. 26, p. 495.

Mathematics in Liberal Education. A Critical Examination of the Judgments of Prominent Men of the Ages. By F. CAJORI. Pp. 169 + viii. $1.50 net. 1928. (The Christopherson Publishing House, Boston, U.S.A.)

According to Prof. Cajori the present position of mathematics in the school curricula of the United States is largely due to the influence of a handful of enthusiasts "at a time when there was a great democratic movement in American secondary education, and when the old ideals of the training of the mind and the aiming at general culture were being displaced by social ideals and vocational tendencies." He sketches the gradual development of a technique in the construction and application of intelligence tests and the recent appearance of considerable scepticism as to the validity of the conclusions based upon processes which are open to the charge of being pedagogically unsound and upon statistics which are numerically insufficient. A remark he makes on one of the results of the experiments on the relative values of various correlations is interesting. Investigation has shown that "algebra and physics is a stronger combination for simultaneous study than algebra and chemistry. Also biology and chemistry is a stronger combination for simultaneous study than biology and physics. But a boy takes all four subjects at the same time. . . . It is evident that the associative law of addition does not hold here." As for "transfer of training," *e.g.* the effective influence of ability in one field upon ability in another, opinions on the whole seem to vary from a downright "absolutely false" to a tepid "non-proven."

Prof. Cajori asks: "Are there not other light-sources available which can be focused upon the fundamental problem of the relative educational value of school subjects?" Referring to the well-known controversy between two famous protagonists, Sir William Hamilton, the metaphysician of Edinburgh, and Whewell, the Master of Trinity, he revives the idea of a general census of opinion on the place of mathematics in education, drawn from the views of the "distinguished men of the ages." But the census must differ from that of Hamilton in not being one-sided. Hamilton's method is described as one of "gathering the judgments of men opposed to mathematics, and disregarding the evidence of those favouring the study." Even more vigorous is the denunciation of Hamilton's attack by Prof. Bledsoe, an American Professor of Mathematics. To crush Whewell "a cloud of witnesses" was produced, but their judgments "are diametrically opposite to those imputed to them by Sir William Hamilton. . . . The slap-dash and reckless way in which he has quoted such authorities, is, perhaps, without parallel in the history of modern literature" (1877). The witnesses summoned by Prof. Cajori are apparently 731 in number (there are about 600 names in the index), of whom 603 "favour mathematics for its superior value in strengthening the mind": if mathematicians are omitted from the list, the count is given as 413 to 123. There are, of course, many objections to this "historical method of attack." In the present case, the most obvious is that among the philosophers, statesmen, educators and mathematicians, whose judgments were consulted," and who "are of necessity, drawn from the distinguished men of the ages," we find men whose claim to be admitted on such a roll of honour is doubtful. We have the opinion of A. S. Johnston, described as "an American Soldier" (he was the Confederate General killed at Shiloh), but we have no reference to Napoleon, whose views on the value of a mathematical training are of some interest. We find Sorbière and Dunton grouped with Swift, Gibbon, and Warburton. Now Sorbière "avait joui quelque temps de la réputation de savant, grâce au talent de saisir les idées des autres dans la conversation, et à l'effronterie de les colporter comme siennes." And Dunton was a bookseller. After failing in business he became a hack-writer and edited a periodical called the *Athenian Mercury*. He was the author of "a curious farrago called *Athenianism* . . . composed with extraordinary self-satisfaction and accompanied with a portrait of the author. This remarkable character died in 1733."

The lists include many such cases, extending from Alpharabius to Edward Adolphus (eleventh Duke of Somerset, who figures in the index as Adolphus, Edward), and makes one wonder on what principle selection of these 700 "distinguished men of the ages" has been made.

However, if we are to count by heads irrespective of brains, the results appear to run as follows : " philosophers and psychologists favor mathematics in liberal education, by a vote of 11 to 6 ; educators by 19 to 14 ; mathematicians by 66 to 3 ; scientists who are not mathematicians by 3 to 1 ; literary men and statesmen by 4 to 1," and business men, lawyers, doctors, and ministers by some 219 to 48.

The author's own contributions to the subject and his general criticisms are sound and temperately expressed. But whether he has been well-advised in devoting his undoubted gifts to an investigation of this type may be fairly questioned.

There is one other point. We must offer to the author our condolences on the way in which his proof-reader has neglected his business. Here are a few suggestions, etc. to which attention may be paid :

P. 36, Pope *Silverter* II, and (f.n.) *Aberlard* : p. 38, Gregory *Thaumarturus* : *Mr.* Descartes (five times on) p. 49 : p. 49, l. 7 up, (he) *strived* ; and l. 5 up, for *touch* read *lead to* : p. 50, l. 10, for *those* read *them as* : p. 51, notes l. 1, *theologish* : p. 53, l. 2, for *met* read *met with* : p. 54, l. 7 up, for *experience* read *experiment* ; notes, l. 2, la *verité* : p. 55, notes l. 1, read *philosophiques* ; and l. 3, read *Intermédiaire* : p. 58, l. 3 up, four acute accents missing : p. 61, Was Fontenelle a mathematician ? ; note l. 1, *Dictionaire* ; and l. 2, *mathematique* : p. 62, note l. 1, *Géometre* should be *Géomètre* or *Géométrie* : p. 63, l. 2, for *principles* read *Principia* : p. 64, note l. 2, *etude* : p. 67, l. 13, for *repulsed* read *repelled* : p. 68, l. 2, delete *for* ; l. 4, add *a manly* (vigor) : p. 69, l. 5, for *rigour* read *vigour* ; l. 10, for *repeated* read *reaped* ; l. 12, *degress* ; p. 70, l. 2, for *such* read *so* : p. 74, l. 19, for *Le Clerk* read *Le Clerc* : p. 75, l. 11 up, *disparge* : p. 78, l. 1, read *academic* (encouragement) ; l. 11, for *to* read *that* : p. 79, l. 5, for *the only* read *almost the only* ; l. 11, *subtelty*, and for *the discipline* read *the best discipline* ; l. 16, after *partial* insert *or feeble* : p. 80, l. 15, for *d'Alembert is* read *Vives and d'Alembert are* : p. 81, l. 19, *contigent* : p. 82, l. 8, *accustomes* ; l. 5 up, *craming* : p. 83, l. 18 ? for *objectional* read *objectionable* : p. 85, l. 9, *simplicty* ; l. 16, Emerson's death 7882 (and Alcuin, p. 35, l. 3, would have been 159 in 894) : p. 86, l. 10, *Sophmore* : p. 92, l. 4, Lacroix was born in 1765 ; l. 15, read *Bézout* for *Bezout* ; note, l. 5 up, for *General* read *général* : p. 95, l. 2, after *excellent* insert *logical* ; l. 2 up, *utterancies* : p. 98, l. 15, *excell* : p. 103, l. 12 up, read *Hermann* : p. 105, l. 9, *remerqué* ; l. 5 up, for *valuate* read *value* : p. 106, l. 12, *the universally potent Leibniz* (!) : p. 112, l. 13, *regreets* : p. 126, l. 8, *demonstrattion* : p. 129, l. 16, *algerbraic* : p. 140, l. 1, for *faits* read *faites* : p. 142, l. 15 up, *Matricualation* : p. 143, l. 4 up, for *to promote* read *of promoting* : p. 148, l. 15, for *deducted* read *deduced* : p. 163, l. 12, after *number* read *of* : p. 169, *irridescent*. Index : (Davy) *Humphry* ; De *Morgan* for *Morgau* ; (Transfer of) *training* ; Sobiere for Sobière (and on pp. 73, 74).

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606. Un vieux capitaine . . . enseignait à ses hommes la manière de s'orienter par l'aspect des étoiles. . . . Il dit donc tout d'abord que les soldats commenceront par chercher dans le ciel l'étoile polaire qui est fixe par rapport aux autres étoiles, lesquelles tournent autour d'elle en sens contraire des aiguilles d'une montre. . . .

" Sacrebleu ! l'abbé, montrez-moi donc cette garce d'étoile polaire. Si je sais la distinguer dans ce fouillis de lumignons dont le ciel est tout semé, je veux que le grand diable me croque ! "

Je lui enseignai incontinent la manière de la trouver et la lui montrai du doigt.

" Oh ! oh ! s'écria-t-il, la pécore est perchée bien haut ! De l'endroit où nous sommes on ne peut la regarder sans se tordre le col."

Et, tout aussitôt, il donna l'ordre à ses officiers de faire reculer les soldats de cinquante pas, pour qu'ils pussent voir plus facilement l'étoile polaire.— Anatole France, *Les Opinions de M. Jérôme Coignard*, cap. xii. [per Mr. F. Puryer White].

607. Condorcet, said d'Alembert, " is a volcano covered with snow." Said another less picturesquely : " he is a sheep in a passion."

YORKSHIRE BRANCH.

THE Summer Meeting of the Yorkshire Branch of the Mathematical Association was held on Saturday, May 5th, at the Wheelwright Grammar School, Dewsbury, when seven new members were elected and 45 members and friends were present. The Chairman of the Branch, W. Peaker, Esq., B.Sc., of Cockburn High School, Leeds, presided over the meeting. Mr. J. M. Child, B.A., B.Sc., Lecturer in Mathematics, Victoria University of Manchester, gave a very instructive paper on "Isaac Barrow and the Calculus"; Miss J. M. Abbott, B.A., Senior Mathematics Mistress at the Girls' High School, Wakefield, gave a most illuminating paper on "The Teaching of Riders in Geometry." Many members entered into the discussion which followed. Mr. L. Sadler, M.A., Head Master of the Wheelwright Grammar School, Dewsbury, kindly entertained all the members and friends to tea. ARTHUR B. OLDFIELD, *Hon. Sec.*

THE NORTH-EASTERN BRANCH.

A MEETING was held on Saturday, May 12th, in Armstrong College for the purpose of inaugurating a Branch of the Mathematical Association; considerable enthusiasm was shown, and more than 60 members were enrolled for the new Branch, which is to be kown as "The North-Eastern Branch." Mr. J. Strachan, H.M.I., was unanimously elected the first President, and after the formal business, gave an address on "Recent Progress in Mathematical Teaching." He sketched the origin and growth of the Association, and showed how it had been largely responsible for what he aptly described as the Geometrical Revolution of 1900-1910. Geometry was now regarded as a conception of knowledge which, like the Natural Sciences, could be reached by more than one approach, and which could not any longer be treated as an ancient philosophy. Mr. Strachan pleaded for more "moving effects" and less rigidity in geometrical teaching, a greater use of graphical methods in the initial stages of Algebra, and the application of trigonometry to certain geometrical exercises. The efforts of the Mathematical Association had led to the Board of Education circulars on Geometry and Algebra, and the effect of the latter had been to abolish the teaching of complicated algebraic computations, so that there was now room in the school curriculum for an early introduction of numerical trigonometry.

Mr. Strachan was thanked for his address on the motion of Dr. Goldsborough. A representative committee was chosen with Mr. Burchnall, of Durham University, as Chairman, and the following were elected Vice-Presidents: Professors Havelock and Heawood, Messrs. Bullerwell, Burchnall, and Caunt, and Dr. Goldsborough.

ERRATUM AND APOLOGY.

Contents: Cover, July 1928, l. 6 up. *For* Daniels, F.R.S., *read* Daniell, Sc.D.

[For the unaccountable mistake in name and style we owe our apologies to Prof. Daniell, a valued contributor.]

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BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

October, 1928.

Bařa, J. *Některé Pokusy o Geometrických Pravděpodobnostech.* [On Geometrical Probabilities.] Pp. 22. n.p. 1928. (Piša, Brno.)

Batchelder, P. M. *An Introduction to Linear Difference Equations.* Pp. viii + 210. 18s. net. (Milford, Ox. Univ. Press : Harvard University Press.)

Bieberbach, L. *Vorlesungen über Algebra.* (4th Edition.) Pp. x + 334. 20 RM. 1928. (Teubner, Leipzig.)

Cajori, F. *Mathematics in Liberal Education.* Pp. 157. $1.50 net. 1928. (Christopher, Boston, U.S.A.)

Campbell, N. R. *An Account of the Principles of Measurement and Calculation.* Pp. ix + 293. 12s. 6d. net. 1928. (Longmans, Green.)

Čech, E. *Propriétés Projectives du Contact.* I. Pp. 35. n.p. 1928. (Piša, Brno.)

Chatley, H. C. *Studies in Molecular Force.* Pp. xi + 118. 7s. 6d. net. 1928. (Griffin.)

Crew, H. *The Rise of Modern Physics.* Pp. xvi + 356. 36 figs. 22s. 6d. 1928. (Baillière, Tindall & Cox.)

Durell, C. V. *Practical School Algebra.* Pp. ix + 320 + lx. Complete, 4s. ; with Answers, 4s. 6d. ; or Part I. 1s. 9d., Part II. 1s. 6d., Part III. 1s. 6d. ; with Answers each part 3d. more. Complete Answers separately, 9d. 1928. (Bell & Sons.)

Fine, H. B. *Calculus.* Pp. viii + 421. 14s. net. 1927. (The Macmillan Co.)

Griffin, F. L. *Mathematical Analysis ; Higher Course.* Pp. x + 512. 8s. 6d. 1928. (Harrap.)

Hilbert, D., and Ackermann, A. *Grundzüge der Theoretischen Logik.* Pp. viii + 120. RM. 7.60. 1928. (Springer, Berlin.)

Hostinsky, B. *Sur les Transformations itérées des Variables Aléatoires.* Pp. 24. n.p. 1928. (Piša, Brno.)

Kennelly, A. E. *Vestiges of Pre-Metric Weights and Measures persisting in Metric-System Europe,* 1926-1927. Pp. xiii + 189. 10s. 6d. 1928. (Macmillan Co.)

Malet, H. *Exposé élémentaire du Calcul Vectoriel et de quelques Applications.* Pp. vii + 69. n.p. 1928. (Gauthier-Villars.)

Padalino, F. Five Sheets of Diagrams illustrating *the Delian Problem, Rectification of Arches, and Trisection of Angles.* n.p. 1928. ("Integrale," 175, 5th Ave., New York, U.S.A.)

Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and their Applications.* Pp. xv + 256 + xxvii. 12s. net. 1928. (Bell & Sons.)

Sahánek, J. *Výklad Vzniku Krátých Elecktromagnetikých vln v Elecktronových Lampách.* II. [Origin of short electro-magnetic waves in valves.] Pp. 24. n.p. 1928. (Piša, Brno.)

Siddons, A. W., and Hughes, R. T. *Trigonometry.* Parts I-III. Pp. viii + 320. 4s. 6d. [Parts I-II, 3s. 6d. ; Part I, 1s. 9d. ; Part II, 2s. 6d. ; Part III, 1s. 9d.] 1928. (Cam. Univ. Press.)

Steward, G. C. *The Symmetrical Optical System.* [Cambridge Tracts, No. 25.] Pp. viii + 102. 7s. 6d. net. 1928. (Cam. Univ. Press.)

Thomson, J. B. *Bell's Every Day Arithmetics.* Pupils' Books, I. and II., 8d. each ; III. and IV., 10d. each. Teacher's Books, with Answers, I. and II., 2s. each ; III. and IV., 2s. 3d. each. 1928. (Bell.)

Thomson, Sir J. J. *Beyond the Electron.* Pp. 43. 2s. 6d. net. 1928. (Cam. Univ. Press.)

Tweedy, A. E. *Junior Geometry.* Pp. 189. 2s. 3d. 1928. (Dent.)

Zahradníček, J. *Měření Termomagnetického Longitudinálního Efektu Potenciálového.* Pp. 19. n.p. 1928. (Piša, Brno.)

American Journal of Mathematics. (Johns Hopkins Press, Baltimore.)

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A Theorem on Correspondences on Algebraic Curves. Pp. 159-166. S. LEFSCHETZ. *Concerning the Structure of a Continuous Curve.* Pp. 167-194. G. T. WHYBURN. *Compound Statements on Four Classes.* Pp. 194-208. F. E. BAKER. *A Memoir on the Invariants of Biternary Quantics.* Pp. 209-230. O. E. GLENN. *Transitive Substitution Groups containing Regular Subgroups of Lower Degree.* Pp. 231-242. F. E. JOHNSTON. *Generalized Green's Functions for compatible Differential Systems.* Pp. 243-258. W. W. ELLIOTT. *An Expansion involving p inseparable Parameters associated with a Partial Differential Equation.* Pp. 259-268. C. C. CAMP. *On the asymptotic Evaluation of Functions defined by Contour Integrals.* Pp. 269-302. D. M. WRINCH. *Generalized Quaternion Algebras and the Theory of Numbers.* Pp. 303-314. L. W. GRIFFITHS.

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Annales de la Société Polonaise de Mathématique. (Drukarnia Universytetu Jagiellonskiego pod Zarz. J. Filipowskiego.) Krakow.

Vol. 6 : 1927, issued June 1928.

Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien. Pp. 1-7. E. CARTAN. *Sur le changement du système de référence pour un champ electromagnétique déterminé.* Pp. 8-49. S. ZAREMBA. *Les ensembles boreliens abstraits.* Pp. 50-53. W. SIERPINSKI. *Sur une classe d'espaces riemanniens à trois dimensions.* Pp. 54-82. W. SLEBODZINSKI. *Un théorème sur les fonctions dérivables.* Pp. 83-92. T. WAZEWSKI. *Sur la loi de probabilité de l'écart maximum.* Pp. 93-116. M. FRÉCHET.

Annals of Mathematics. (Princeton University Press, N.J.)

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Anuario. (Universidad Nacional de La Plata.)

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Boletin Matematico. (Dr. Baidaff, Belgrano 909, Buenos Aires.)

Vol. 1 : No. 1, Jan. 15, 1928.

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Enlazar por interpolacion puntos, curvas, superficies, etc. Pp. 5-6. B. I. BAIDAFF.

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Enlace por interpolacion de Figuras de distintas dimensiones. Pp. 9-10. B. I. BAIDAFF.

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Bollettino della Unione Matematica Italiana. (Zanichelli, Bologna.)

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Bulletin of the Calcutta Mathematical Society. (Calcutta Univ. Press.)

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Contribución al Estudio de las Ciencias Físicas y Matemáticas. (Universidad Nacional de la Plata.)

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Perhaps readers *who find the analyses of definite value* will be so good as to send to the Editor a card with their names and the words : Analyses appreciated.

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LONDON BRANCH.

PROGRAMME FOR 1928-29.

All meetings will be held at Bedford College, Regent's Park.
They will be on Saturday afternoons, and will begin at 3 p.m.

1928.

Nov. 10th. "What should we teach in Graphs ?"—C. T. Daltry.

Dec. 8th. "How do we justify the place that Mathematics has hitherto held in the School Curriculum ?"—J. Katz.

1929.

Feb. 2nd. Presidential Address : "Dimensions and Identity of Vector direction."—Professor Alfred Lodge.

Feb. 23rd. Annual Business Meeting.
Discussion of Members' Questions.

Mar. 16th. Debate on Standard Form.
(Speakers : Miss Punnett and A. S. Grant.)

The Secretaries will be glad to receive :

(i) Topics for discussion on February 23rd.
(ii) Names of Members willing to take part in the Debate on March 16th.

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